

High- T_c Superconductivity via BCS and BEC Unification: A Review

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Abstract

Efforts to unify the Bardeen, Cooper & Schrieffer (BCS) and the Bose-Einstein condensation (BEC) formalisms in terms of a “*complete* boson-fermion (BF) model” (CBFM) are surveyed. A vital distinction is that Cooper pairs (CPs) are indeed bosons that suffer BEC, in contrast with BCS pairs that are not bosons. Another crucial ingredient (particularly in 2D where ordinary BEC does not occur) is the *linear* dispersion relation of “ordinary” CPs, at least in leading order in the center-of-mass momentum (CMM) power-series expansion of the CP energy. This arises because CPs propagate not *in vacuo* but in the Fermi “sea.” A many-body Bethe-Salpeter equation treatment of CPs based on the ideal Fermi gas (IFG) sea yields the familiar negative-energy, two-particle bound-state *if* 2h-CPs are ignored as in the ordinary CP problem. But it gives purely-imaginary energies, and is thus meaningless, if 2h-CPs are included as completeness requires. However, when based on the BCS ground state instead of the IFG, in addition to the familiar trivial solution (or Anderson-Bogoliubov-Higgs) sound mode, legitimate two-particle *moving* “generalized CPs” emerge but as positive-energy, finite-lifetime, resonant nontrivial solutions for nonzero-CMM. This amounts to replacing the purely-kinetic-energy unperturbed Hamiltonian by the BCS one. The moving CPs again have a *linear* dispersion leading term. BEC of such pairs may thus occur in exactly 2D (as it cannot with quadratic dispersion) and in fact all the way down to $(1 + \epsilon)D$ where ϵ can be infinitesimally small, thus encompassing all empirically known superconductors.

The unified theory reduces in limiting cases to all the main continuum (as opposed to “spin”) statistical theories of superconductivity. These include both the BCS and BEC theories. The unified BF theory is “complete” in that not only two-electron (2e) but also two-hole (2h) CPs are allowed, and in arbitrary proportions. In contrast, BCS theory can be deduced from the CBFM but allows only equal (50%-50%) mixtures of them, a fact rarely if ever stressed. The CBFM shows that the BCS condensate is precisely a BE condensate of a mixture of kinematically independent electrons coexisting with weakly-coupled zero CMM 2e- and 2h-CPs in *equal* proportions. Without abandoning the electron-phonon mechanism, the CBFM has been applied in 2D and 3D. The BCS model interaction in moderately weak coupling is sufficient to reproduce the unusually high values of T_c (in units of the Fermi temperature) of $0.01 - 0.1$ empirically exhibited by the so-called “exotic” superconductors, including cuprates. This range is high relative to the low values of $\leq 10^{-3}$ more or less correctly reproduced by BCS theory for conventional (mostly elemental) superconductors. Also accounted for is the empirical fact that “hole superconductors” systematically have higher T_c ’s. Room temperature superconductors are predicted to be possible but only via 2h-CP BE condensates.

Running Title: BCS and BEC Unification: A Review

1 Introduction

Boson-fermion (BF) models of superconductivity (SC) as a Bose-Einstein condensation (BEC) [1][2] go back to the mid-1950's [3]-[6], pre-dating even the BCS-Bogoliubov theory [7]-[9]. Although BCS theory only contemplates the presence of “Cooper correlations” of single-particle states, BF models [3]-[6],[10]-[18] posit the existence of actual bosonic CPs. Such pair charge carriers have been observed in magnetic flux quantization experiments on elemental [19, 20] as well as on cuprate [21] superconductors (SCs). Larger clusters than pairs are *not* observed, apparently because the clustering occurs not *in vacuo* but in the Fermi sea. However, no experiment has yet been done, to our knowledge, that distinguishes between electron and hole CPs. CPs appear to be the single most important universally accepted ingredient of SC, whether conventional or “exotic” and whether of low- or high-transition-temperatures T_c . And yet, inspite of their centrality they are poorly understood. The fundamental drawback of early [3]-[6] BF models, which took 2e bosons as analogous to diatomic molecules in a classical atom-molecule gas mixture, is the notorious absence of an electron energy gap $\Delta(T)$. “Gapless” models cannot describe the superconducting state at all, although they are useful in locating transition temperatures if approached from above, i.e., $T > T_c$. Even so, we are not aware of any calculations with the early BF models attempting to reproduce any empirical T_c values. The gap first began to appear in later BF models [10]-[15]. With two [13][14] exceptions, however, all BF models neglect the effect of *hole* CPs included on an equal footing with electron CPs to give a “complete” BF model (CBFM) consisting of *both* bosonic CP species coexisting with unpaired electrons.

For 13 years the highest T_c value for any superconductor was 23 K, until the discovery [22] in 1986 of the first so-called “high- T_c ” cuprate superconductor *LaBaCuO* having a $T_c \simeq 35$ K. A feverish search for materials with even higher T_c 's lead within just seven years to the highest- T_c superconductor known to date, the *HgBaCaCuO* cuprate [23] with a $T_c \simeq 164$ K under very high pressure ($\simeq 310,000$ atm). The embarrassing fact is that since 1993 this record has not been broken, very probably because there is yet no *predictive* microscopic theory of superconductivity that can provide orientation in a search that until now has proceeded on a trial-and-error basis.

We submit that progress in developing such a theory has been held back by too many common myths or false dogmas firmly entrenched in the theoretical community.

2 Some false dogmas

The false theoretical dogmas just mentioned can be summarized in the following assertions:

1. With the electron-phonon dynamical mechanism transition temperatures (as given by the BCS formula) $T_c = 1.13\Theta_D e^{-1/\lambda} \lesssim 45$ K at most, since a typical Debye temperature $\Theta_D \sim 300$ K and $\lambda \lesssim \frac{1}{2}$. For higher T_c 's one needs to invoke magnons or excitons or plasmons or other electronic mechanisms to provide pairing [24].
2. Superconductivity is unrelated to Bose-Einstein condensation (BEC) [25].
3. BEC is impossible in 2D [26][27].
4. Cooper pairs:
 - a) are such that “there is a very strong preference for singlet, zero-momentum pairs, so strong that one can get an adequate description of SC by treating these correlations alone [28].”
 - b) consist of negative-energy stable (i.e., stationary) bound states [29].
 - c) propagate in the Fermi sea with an energy $\hbar^2 K^2 / 2(2m)$, where m is the effective electron (or hole) mass and $\hbar K$ the CP center-of-mass momentum (CMM) [30], hence assertion # 3 above.

- d) with the linear dispersion $E \propto v_F \hbar K$, where v_F is the Fermi velocity, simply represent the sound mode of the ideal Fermi gas (+ interactions), with sound speed v_F/\sqrt{d} in any dimensionality d [31].
- e) “...with $K \neq 0$ represent states with net current flow” [32].
- f) and BCS pairs are the same thing [33].
- g) are *not* bosons, Ref. [34] p. 38, hence assertion # 2 above.

3 Ordinary Cooper pairing

For bosons with excitation (e.g., kinetic) energy for small CMM K given by

$$\varepsilon_K = C_s K^s + o(K^s), \quad (1)$$

with C_s some coefficient and $s > 0$, BEC occurs in a box of length L if and only if $d > s$, since $T_c \equiv 0$ for all $d \leq s$. The commonest example is $s = 2$ as in the textbook case of ordinary bosons with exactly $\varepsilon_K = \hbar^2 K^2/2m_B$, with m_B the boson mass, giving the familiar result that BEC is not allowed for $d \leq 2$. The general theorem for any $s > 0$ is stated as follows. The total boson number is

$$N = N_0(T) + \sum_{\mathbf{K} \neq \mathbf{0}} [\exp \beta(\varepsilon_K - \mu_B) - 1]^{-1} \quad (2)$$

with $\beta \equiv k_B T$. Since $N_0(T_c) \simeq 0$ while the boson chemical potential μ_B also vanishes at $T = T_c$, in the thermodynamic limit the (finite) boson number density becomes

$$N/L^d \simeq A_d \int_{0+}^{\infty} dK K^{d-1} [\exp \beta_c (C_s K^s + \dots) - 1]^{-1} \quad (3)$$

where A_d is a finite coefficient. Thus

$$N/L^d \simeq A_d (k_B T_c / C_s) \int_{0+}^{K_{\max}} dK K^{d-s-1} + \int_{K_{\max}}^{\infty} \dots, \quad (4)$$

where K_{\max} is small and can be picked arbitrarily so long as the integral $\int_{K_{\max}}^{\infty} \dots$ is finite, as is N/L^d . However, if $d = s$ the first integral gives $\ln K \big|_0^{K_{\max}} = -\infty$; and if $d < s$ it gives $1/(d-s) K^{s-d} \big|_0^{K_{\max}} = -\infty$. Hence, T_c must vanish if and only if $d \leq s$, but is otherwise finite. This conclusion hinges *only* on the leading term of the boson dispersion relation ε_K .

The case $s = 1$ emerges in both the “ordinary” CP problem [29] to be recalled now, as well as in the “generalized” case of the next section. The question of whether or not CPs are bosons or not is resolved in Appendix A. For $s = 1$ BEC occurs for all $d > 1$. Striking experimental confirmation of how superconductivity is “extinguished” as dimensionality d is diminished towards unity has been reported by Tinkham and co-workers [35][36]. They report conductance *vs.* diameter curves in superconducting nanowires consisting of carbon nanotubes sputtered with amorphous $Mo_{79}Ge_{21}$ ($T_c \simeq 5.5$ K) and of widths from 22 to 10 nm. The conductance diminishes to zero, implying the nanotube becomes an insulator below a certain diameter, thus exposing how T_c vanishes for the thinnest nanotubes.

The CP equation for the energy \mathcal{E}_K of two fermions above the Fermi surface with momentum wavevectors \mathbf{k}_1 and \mathbf{k}_2 (and arbitrary CMM wavenumber K where $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$) is given by

$$[\hbar^2 k^2/m - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m] \psi_{\mathbf{k}} = - \sum_{\mathbf{q}}' V_{\mathbf{kq}} \psi_{\mathbf{q}}, \quad (5)$$

where $\mathbf{k} \equiv \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$ is the CP relative momentum and $\psi_{\mathbf{q}}$ its wave function in momentum space. The prime on the summation implies restriction to states *above* the Fermi surface with energy $E_F \equiv \hbar^2 k_F^2/2m$, viz., $|\mathbf{k} \pm \mathbf{K}/2| > k_F$, and $V_{\mathbf{kq}}$ is the double Fourier transform of the interaction defined as

$$V_{\mathbf{kq}} \equiv \frac{1}{L^d} \int d\mathbf{r} \int d\mathbf{r}' e^{-i\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}, \mathbf{r}') e^{i\mathbf{k}\cdot\mathbf{r}'}, \quad (6)$$

with $V(\mathbf{r}, \mathbf{r}')$ the (possibly nonlocal) interaction in real d -dimensional space.

3.1 Delta interaction

If the interfermion interaction $V(\mathbf{r}, \mathbf{r}')$ is local, then $V(\mathbf{r}, \mathbf{r}') = V(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}')$ in (6). Moreover, if $V(r) = -v_0\delta(\mathbf{r})$ with $v_0 > 0$, this gives $V_{\mathbf{kq}} = v_0/L^d$ and (5) becomes, for any d ,

$$\frac{1}{L^d} \sum'_{\mathbf{k}} \frac{1}{\hbar^2 k^2/m - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m} = \frac{1}{v_0}. \quad (7)$$

In $d = 1$ where the δ -well supports a single bound state, the problem is quite tractable [37]. In either 2D or 3D, however, the δ -well supports an infinite set of bound levels with the lowest level in each case being infinitely bound. This in turn leads to a rigorous collapse of the many-fermion system [38]. To prevent this unphysical collapse the δ -wells must be “regularized,” i.e., constructed, say, from square wells [39] such that the remaining δ -well possesses only one bound level. This leaves an infinitesimally small strength parameter v_0 which would make the rhs of (7) diverge (so as to cancel the lhs that also diverges in 2D and 3D but not in 1D). Combining (7) for 2D with the vacuum two-body momentum-space Schrödinger equation for the same δ -potential well allows eliminating [40] v_0 in favor of the (positive) binding energy B_2 of the single bound level of the regularized δ -well. One arrives at

$$\sum_{\mathbf{k}} \frac{1}{B_2 + \hbar^2 k^2/m} = \sum'_{\mathbf{k}} \frac{1}{\hbar^2 k^2/m - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m}, \quad (8)$$

where $B_2 \geq 0$ now serves as a coupling constant. A small- K power-series expansion for \mathcal{E}_K gives the analytic expression *valid for any dimensionless coupling* $B_2/E_F \geq 0$,

$$\varepsilon_K \equiv \mathcal{E}_K - \mathcal{E}_0 = \frac{2}{\pi} \hbar v_F K + \left[1 - (2 - [4/\pi]^2) \frac{E_F}{B_2} \right] \frac{\hbar^2 K^2}{2(2m)} + O(K^3), \quad (9)$$

where a nonnegative *CP excitation energy* ε_K has been defined, and the Fermi velocity v_F comes from $E_F/k_F = \hbar v_F/2$. The leading term in (9) is linear in K , followed by a quadratic term. It is clear that the leading term in (9) is quadratic, namely

$$\varepsilon_K = \hbar^2 K^2/2(2m) + O(K^3), \quad (10)$$

provided v_F and hence E_F vanish, i.e., there is no Fermi sea. This is just the familiar nonrelativistic kinetic energy in vacuum of the composite (so-called “local”) pair of mass $2m$ and CMM K . The same result (10) is also found to hold in 3D but not analytically as here in 2D. Figure 1 shows exact numerical results of a dimensionless CP excitation energy $\varepsilon_K/(-\mathcal{E}_0)$ (in figure, Δ_0 means present $-\mathcal{E}_0 > 0$) as function of K/k_F for different couplings B_2 . Note that ordinary CPs *break up* whenever \mathcal{E}_K turns from negative to positive, i.e., when \mathcal{E}_K vanishes, or by (9) when $\varepsilon_K/(-\mathcal{E}_0) \equiv \varepsilon_K/\Delta_0 = 1$. These points are marked in the figure by dots. In addition to the exact results (full curves) also shown are some results for the linear approximation [first term on the right-hand side of (9), dot-dashed lines (virtually coinciding with the exact curve for all $B_2/E_F \lesssim 0.1$)], as well as for the quadratic approximation (dashed parabolas) as given by the leading term in (10) for stronger couplings. For weak enough coupling the exact dispersion relation is virtually linear—in spite of the divergence

of the quadratic term in (9) as $B_2/E_F \rightarrow 0$. As E_F decreases the quadratic dispersion relation (10) very slowly begins to dominate. A result unique to 2D (and associated with the fact that in 2D the fermionic density of states is *independent* of energy) is that $-\mathcal{E}_0 \equiv \Delta_0 = B_2$.

In 3D [41] instead of (8) similar procedures for two-component fermions give

$$\sum_{\mathbf{k}} \frac{1}{\hbar^2 k^2/m} - \sum_{\mathbf{k}, (|\mathbf{k} \pm \mathbf{K}/2| > k_F)} \frac{1}{\hbar^2 k^2/m - \mathcal{E}_K - 2E_F + \hbar^2 K^2/4m} = \frac{mL^3}{4\pi\hbar^2} \frac{1}{a} \quad (11)$$

where a is the s -wave scattering length associated with the regularized δ -well, which corresponds with weak to strong coupling according as $-\infty < 1/k_F a < +\infty$. One finds for weak coupling ($k_F a \rightarrow 0^-$, e.g., prior to the well-known first-bound-state singularity as the depth of a 3D potential well is increased) that

$$\mathcal{E}_0/E_F \rightarrow -(8/e^2) \exp(-\pi/k_F |a|), \quad (12)$$

a result first reported by Van Hove [42]. For strong coupling ($k_F a \rightarrow 0^+$, beyond the single-bound-state resonance) one gets

$$\mathcal{E}_0/E_F \rightarrow -2/(k_F a)^2. \quad (13)$$

Numerical results in 3D very similar to those in Fig. 1 for 2D are obtained. Namely, for weak coupling the CP dispersion curves are very nearly linear while for smaller density they very slowly tend to the quadratic. The limit given by (9) in 2D was found too complicated in 3D to be evaluated analytically, except for weak coupling. Repeating the 2D analysis without attempting to explicitly determine the coefficient of the quadratic term one gets, for the δ -well interfermion interaction in weak coupling,

$$\varepsilon_K \equiv \mathcal{E}_K - \mathcal{E}_0 = \frac{1}{2} \hbar v_F K + O(K^2). \quad (14)$$

This is the same result cited without proof by Schrieffer in 1964 (Ref. [34], p. 33) for the BCS model interaction, to which we now turn.

3.2 BCS model interaction

The BCS model interaction is of the simple form

$$V_{\mathbf{k}\mathbf{k}'} = \begin{cases} -V & \text{if } k_F < |\mathbf{k} \pm \frac{1}{2}\mathbf{K}|, |\mathbf{k}' \pm \frac{1}{2}\mathbf{K}| < \sqrt{k_F^2 + k_D^2} \\ 0 & \text{otherwise,} \end{cases} \quad (15)$$

where $V_{\mathbf{k}\mathbf{k}'}$ is defined in (6). Here $V > 0$, and $\hbar\omega_D \equiv \hbar^2 k_D^2/2m$ is the maximum energy of a vibrating-ionic-lattice phonon. This means that two fermions interact with a constant attraction $-V$ when the tip of their relative-momentum wavevector \mathbf{k} points anywhere inside the overlap volume in k -space of the two spherical shells in Figure 2. Inserting (15) into (5) and converting sums over \mathbf{k} into energy integrals by introducing the electronic density of states (DOS) $g(\epsilon)$ gives

$$1 = V \sum_{\mathbf{k}}' [2\epsilon_k - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m]^{-1} = V \int_{E_F}^{E_F + \hbar\omega_D} \frac{g(\epsilon)d\epsilon}{2\epsilon - 2E_F - \mathcal{E}_K + \hbar^2 K^2/4m}. \quad (16)$$

From this one immediately obtains the familiar result for $K = 0$,

$$\mathcal{E}_0 = -\frac{2\hbar\omega_D}{e^{2/\lambda} - 1} \xrightarrow{\lambda \rightarrow 0} -2\hbar\omega_D e^{-2/\lambda}. \quad (17)$$

Here $\lambda \equiv g(E_F)V$ is a dimensionless coupling constant and $g(E_F)$ the DOS for each spin evaluated at the Fermi energy. The equality in (17) is *exact* in 2D for all coupling—as well as in 1D or 3D provided only that $\hbar\omega_D \ll E_F$ so that $g(\epsilon) \simeq g(E_F)$, a constant that can be taken outside the integral in (16).

For a 2D system (16) gives [43] for weak coupling

$$\mathcal{E}_K \xrightarrow{K \rightarrow 0} \mathcal{E}_0 + (2/\pi)\hbar v_F K + O(K^2). \quad (18)$$

The exact dispersion relation obtained numerically from (16) for $\lambda = \frac{1}{2}$ and $\hbar\omega_D/E_F = 10^{-2}$ shows that the linear approximation (18) is very good for moderately small λ and $\hbar\omega_D/E_F$, over the entire range of K values for which $\mathcal{E}_K \leq 0$. Note that the linear term carries the *same* coefficient as (9) for a *different* interfermion interaction. Pair breakup, specifically $\mathcal{E}_K > 0$ for these values of λ and $\hbar\omega_D/E_F$, occurs at a relatively small value of K , about four orders of magnitude smaller than the maximum value $2\sqrt{k_F^2 + k_D^2}$ allowed by the interaction (15).

In 3D, assuming $\hbar\omega_D/E_F \ll 1$ so that the 3D DOS $g(\varepsilon) = (L^3/\pi^2\hbar^3)\sqrt{m^3\varepsilon/2}$ can be replaced by $g(E_F)$ and then taken outside the integral sign, the result cited in Ref. [34], p. 33, (see also Ref. [44], p. 336, Prob. 10.4 but note here an erroneous coefficient) follows, namely

$$\mathcal{E}_K \xrightarrow{K \rightarrow 0} \mathcal{E}_0 + \frac{1}{2}\hbar v_F K + O(K^2). \quad (19)$$

Exact numerical results in 3D are qualitatively similar [43] to those in 2D as regards goodness of the linear approximation for weak coupling.

4 Generalized Cooper pairing

The “ordinary” CP problem just summarized for two distinct interfermion interactions (the δ -well and the BCS model interaction) neglects the effect of two-hole (2h) CPs treated on an equal footing with two-particle (2p) CPs—as Green’s functions [44] can naturally guarantee. On the other hand, the BCS condensate consists [13] of *equal numbers of 2p and 2h Cooper “correlations.”* This was already evident, though scarcely emphasized, from the perfect symmetry about μ , the electron chemical potential, of the well-known Bogoliubov [45] $v^2(\epsilon)$ and $u^2(\epsilon)$ coefficients [see just below (30) later on], where ϵ is the electron energy. the prime motivation here rests on the recently established remarkable fact [13] that a BCS condensate is precisely a BE condensate with equal numbers of 2p and 2h zero-CMM CPs, in the limit of weak coupling. Further motivation comes from the unique but unexplained role played by *hole* charge carriers in the normal state of superconductors [46] in general (see also Ref. [47]). Final motivation stems from the ability of the “complete (in that both 2h- and 2p-CPs are allowed in varying proportions) BF model” of Refs. [13]-[15] to “unify” both BCS and BEC theories as special cases, and to predict substantially higher T_c ’s than BCS theory without abandoning electron-phonon dynamics. The latter is important as compelling evidence for a significant, if not sole, presence of it in high- T_c cuprate superconductors from angle-resolved photoemission spectroscopy data has recently been reported [48].

In this section we sketch how the Bethe-Salpeter (BS) many-body equation (in the ladder approximation) treating both 2p and 2h pairs on an equal footing shows that, while the ordinary CP problem [based on an ideal Fermi gas (IFG) ground state (the usual “Fermi sea”)] does *not* possess stable energy solutions, but that it does so when the IFG ground state is replaced by the BCS one. This is equivalent to starting from an unperturbed Hamiltonian that is the BCS ground state instead of the pure-kinetic-energy operator corresponding to the IFG. We discuss how: i) CPs based not on the IFG-sea but on the BCS ground state survive in a *nontrivial* solution as “generalized” or “moving” CPs which are *positive* energy resonances with an imaginary energy term leading to finite-lifetime effects; ii) as in the “ordinary” CP problem of the previous section, their dispersion relation in leading order in the total (or center-of-mass) momentum (CMM) $\hbar\mathbf{K} \equiv \hbar(\mathbf{k}_1 + \mathbf{k}_2)$ is also *linear* rather than the quadratic $\hbar^2 K^2/2(2m)$ of a composite boson (e.g., a deuteron) of mass $2m$ moving not in the Fermi sea but in vacuum; and iii) this latter “moving CP” solution, though often confused [31] with it, is

physically *distinct* from another more common *trivial* solution sometimes called, even though Bogoliubov [8] was the first to derive it, the Anderson-Bogoliubov-Higgs (ABH) [49], ([9] p. 44), [50]-[52] collective excitation. The ABH mode is also linear in leading order and goes over into the IFG ordinary sound mode in zero coupling. All this occurs in both 2D [53] as well as in the 3D study outlined earlier in Ref. [54]. We focus here on 2D because of its interest [55]-[57] for quasi-2D high- T_c cuprate superconductors. In general, the results will be crucial for BEC scenarios employing BF models of superconductivity, not only *in exactly 2D* as with the Berezinskii-Kosterlitz-Thouless [58][59] transition, but also down to $(1 + \epsilon)\text{D}$ which characterize the quasi-1D organo-metallic (Bechgaard salt) superconductors [60]-[62]. These results also apply, albeit with a different interaction, to neutral-atom superfluidity as in liquid ^3He [63] and very probably also in ultracold trapped alkali Fermi gases such as ^{40}K [64] and ^6Li [65] atoms.

In dealing with the many-electron system we again assume the BCS model interaction (6) in 2D with double Fourier transform

$$\begin{aligned}\nu(k_1, k'_1) &= -(k_F^2/k_1 k'_1)V \quad \text{if } k_F - k_D < k_1, k'_1 < k_F + k_D, \\ &= 0 \quad \text{otherwise.}\end{aligned}\tag{20}$$

Here $V > 0$, $\hbar k_F \equiv m v_F$ the Fermi momentum, m the effective electron mass, v_F the Fermi velocity, and $k_D \equiv \omega_D/v_F$ with ω_D the Debye frequency; note difference with previous definition just below (15). The usual physical constraint $\hbar\omega_D \ll E_F$ then implies that $k_D/k_F \equiv \hbar\omega_D/2E_F \ll 1$. Assuming perfect ph symmetry about the Fermi surface, we set

$$\epsilon_k \simeq E_F + \hbar v_F(k - k_F)\tag{21}$$

as it simplifies all calculations when very near the Fermi surface.

The bound-state BS wavefunction equation [54] in the ladder approximation with both particles and holes for the original IFG-based CP problem is

$$\begin{aligned}\Psi(\mathbf{k}, E) = & -\left(\frac{i}{\hbar}\right)^2 G_0(\mathbf{K}/2 + \mathbf{k}, \mathcal{E}_K/2 + E) G_0(\mathbf{K}/2 - \mathbf{k}, \mathcal{E}_K/2 - E) \times \\ & \times \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dE' \frac{1}{L^d} \sum_{\mathbf{k}'} v(|\mathbf{k} - \mathbf{k}'|) \Psi(\mathbf{k}', E').\end{aligned}\tag{22}$$

Here L^d is the “volume” of the d -dimensional system; $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$ is the CMM and $\mathbf{k} \equiv \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$ the relative wavevectors of the 2e bound state whose wavefunction is $\Psi(\mathbf{k}, E)$; $\mathcal{E}_K \equiv E_1 + E_2$ is the energy of this bound state while $E \equiv E_1 - E_2$, and $G_0(\mathbf{K}/2 + \mathbf{k}, \mathcal{E}/2 + E)$ is the bare one-fermion Green’s function given by

$$G_0(\mathbf{k}_1, E_1) = \frac{\hbar}{i} \left\{ \frac{\theta(k_1 - k_F)}{-E_1 + \epsilon_{\mathbf{k}_1} - E_F - i\varepsilon} + \frac{\theta(k_F - k_1)}{-E_1 + \epsilon_{\mathbf{k}_1} - E_F + i\varepsilon} \right\}\tag{23}$$

where $\epsilon_{\mathbf{k}_1} \equiv \hbar^2 k_1^2/2m$ and $\theta(x) = 1$ for $x > 0$ and $= 0$ for $x < 0$, so that the first term refers to *electrons* and the second to *holes*. Figure 3 shows all Feynman diagrams for the 2p, 2h and ph wavefunctions ψ_+ , ψ_- and ψ_0 , respectively, that emerge in the general (BCS-ground-state-based) problem to be discussed later. For the present IFG-based case, diagrams in shaded rectangles do *not* contribute as they involve factors of $\theta(k_1 - k_F)\theta(k_F - k_1) \equiv 0$. Since the energy dependence of $\Psi(\mathbf{k}, E)$ in (22) is only through the Green’s functions, the ensuing energy integrals (22) can be evaluated directly in the complex E' -plane and yield, for interaction (20), an equation for the wavefunction $\psi_{\mathbf{k}}$ in momentum space for CPs with *zero* CMM $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2 = 0$ that is

$$(2\xi_k - \mathcal{E}_0)\psi_{\mathbf{k}} = V \sum'_{\mathbf{k}'} \psi_{\mathbf{k}'} - V \sum''_{\mathbf{k}'} \psi_{\mathbf{k}'}.\tag{24}$$

Here $\xi_k \equiv \hbar^2 k^2/2m - E_F$ while \mathcal{E}_0 is the $\mathbf{K} = 0$ eigenvalue energy, and $\mathbf{k} \equiv \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2) = \mathbf{k}_1$. The single prime over the first (2p-CP) summation term denotes the restriction $0 < \xi_{k'} < \hbar\omega_D$ while the double prime

in the last (2h-CP) term means $-\hbar\omega_D < \xi_{k'} < 0$. Without this latter term we have Cooper's Schrödinger-like equation [29] for 2p-CPs whose implicit solution is clearly $\psi_{\mathbf{k}} = (2\xi_k - \mathcal{E}_0)^{-1} V \sum_{\mathbf{k}'} \psi_{\mathbf{k}'}$. Since the summation term is constant, performing that summation on both sides allows canceling the $\psi_{\mathbf{k}}$ -dependent terms, leaving the eigenvalue equation $\sum_{\mathbf{k}}' (2\xi_k - \mathcal{E}_0)^{-1} = 1/V$. This is one equation in one unknown \mathcal{E}_0 ; transforming the sum to an integral over energies gives the familiar solution (17) exact in 2D, and to a very good approximation otherwise if $\hbar\omega_D \ll E_F$, where $\lambda \equiv VN(E_F)$ with $N(E_F)$ the electronic DOS for one spin. This corresponds to a negative-energy, stationary-state bound pair. For $K \geq 0$ the 2p-CP eigenvalue equation becomes

$$\sum_{\mathbf{k}}' (2\xi_k - \mathcal{E}_K + \hbar^2 K^2/4m)^{-1} = 1/V. \quad (25)$$

Note that a 2p CP state of energy \mathcal{E}_K is characterized only by a definite K but *not* definite \mathbf{k} , in contrast to a “BCS pair” defined [Ref. [7], Eqs. (2.11) to (2.13)] with fixed \mathbf{K} and \mathbf{k} (or equivalently definite \mathbf{k}_1 and \mathbf{k}_2); see Appendix A. Without the first summation term in (24) the same expression (17) for \mathcal{E}_0 for 2p-CPs follows for 2h-CPs, apart from an overall sign change.

The *complete* equation (24) *cannot* be derived from an ordinary (non-BS) Schrödinger-like equation in spite of its simple appearance. To solve it for the unknown energy \mathcal{E}_0 , let the rhs of (24) be defined as $A - B$, with A relating to the 2p term and B to the 2h one. Solving for the unknown $\psi_{\mathbf{k}}$ gives

$$\psi_{\mathbf{k}} = (A - B)/(2\xi_k - \mathcal{E}_0) \quad \text{or equivalently} \quad \psi(\xi) = (A - B)/(2\xi - \mathcal{E}_0) \quad (26)$$

whence

$$A \equiv \lambda \int_0^{\hbar\omega_D} d\xi \psi(\xi) = \frac{1}{2}(A - B)\lambda \int_{-\mathcal{E}_0}^{2\hbar\omega_D - \mathcal{E}_0} dz/z \equiv (A - B)x \quad (27)$$

$$B \equiv \lambda \int_{-\hbar\omega_D}^0 d\xi \psi(\xi) = \frac{1}{2}(A - B)\lambda \int_{-2\hbar\omega_D - \mathcal{E}_0}^{-\mathcal{E}_0} dz/z \equiv (A - B)y. \quad (28)$$

The integrals are readily evaluated giving $x \equiv \frac{1}{2}\lambda \ln(1 - 2\hbar\omega_D/\mathcal{E}_0)$ and $y \equiv -\frac{1}{2}\lambda \ln(1 + 2\hbar\omega_D/\mathcal{E}_0)$. As A and B still contain the unknown $\psi(\xi)$ let us eliminate them. Note that (27) and (28) are equivalent to *two* equations in two unknowns A and B , namely

$$\begin{aligned} (1 - x)A + xB &= 0 \\ -yA + (1 + y)B &= 0. \end{aligned}$$

This leads immediately to the equation $1 - x + y = 0$, which on inserting the definitions of x and y becomes

$$1 = \frac{1}{2}\lambda \ln[1 - (2\hbar\omega_D/\mathcal{E}_0)^2],$$

which finally yields

$$\mathcal{E}_0 = \pm i2\hbar\omega_D/\sqrt{e^{2/\lambda} - 1}. \quad (29)$$

As the CP energy is pure-imaginary there is an obvious instability of the CP problem when both particle- and hole-pairs are included. This was reported in Refs. [9] p. 44 and [66], who did not, however, stress the pure 2p and 2h special cases just discussed. Clearly then, the original CP picture *is meaningless if particle- and hole-pairs are treated on an equal footing*, as consistency demands.

5 BCS-based BS treatment of Cooper pairing

However, a BS treatment not about the IFG sea but about the BCS ground state *vindicates the CP concept* as a nontrivial solution. This is equivalent to starting not from the IFG unperturbed Hamiltonian but from the

BCS one. Its physical justification lies in recovering three expected items: ABH sound mode, the BCS $T = 0$ gap equation and finite-lifetime effects of the “moving CPs.” In either 2D [53] or 3D [54] the BS equation yields a 4×4 determinant which reduces to a 3×3 and a 1×1 determinant representing, respectively, *two distinct solutions*: a) the trivial ABH sound solution and b) a highly *nontrivial* moving CP solution, respectively. In either case the BS formalism gives a set of three coupled equations, one for each (2p, 2h and ph) channel wavefunction for any spin-independent interaction such as (20). However, the ph channel decouples, leaving only two coupled wavefunction equations for the ABH solution in 2D which we consider first. We note that in Ref. [67] the hh channel was ignored, leading to a 3×3 determinant from which only the trivial ABH solution emerges, but the nontrivial moving CP one was missed entirely. Thus, the IFG Green function (23) is replaced by the BCS one

$$\mathbf{G}_0(\mathbf{k}_1, E_1) = \frac{\hbar}{i} \left\{ \frac{v_{k_1}^2}{-E_1 + E_{k_1} - i\varepsilon} + \frac{u_{k_1}^2}{-E_1 + E_{k_1} + i\varepsilon} \right\} \quad (30)$$

where $E_{\mathbf{k}} \equiv \sqrt{\xi_{\mathbf{k}}^2 + \Delta^2}$ with Δ the $T = 0$ fermionic gap, $v_k^2 \equiv \frac{1}{2}(1 - \xi_k/E_{\mathbf{k}})$ and $u_k^2 \equiv 1 - v_k^2$ are the Bogoliubov functions [68]. As $\Delta \rightarrow 0$ these three quantities become $|\xi_k|$, $\theta(k_1 - k_F)$ and $\theta(k_F - k_1)$, respectively, so that (30) reduces to (23), as expected. Substituting $G_0(\mathbf{k}_1, E_1)$ by $\mathbf{G}_0(\mathbf{k}_1, E_1)$ corresponds to rewriting the total Hamiltonian so that the pure-kinetic-energy unperturbed Hamiltonian is replaced by the BCS one. The remaining Hamiltonian terms are then assumed suitable to a perturbation treatment. We focus in this section only on 2D.

5.1 ABH sound (trivial) solution

The equations involved are too lengthy even in 2D and will be derived in detail elsewhere, but for the trivial ABH sound solution the aforementioned 3×3 determinant boils down to the single expression

$$\begin{aligned} & \frac{1}{2\pi} \lambda \hbar v_F \int_{k_F - k_D}^{k_F + k_D} dk \int_0^{2\pi} d\varphi \{ u_{\mathbf{K}/2+\mathbf{k}} u_{\mathbf{K}/2-\mathbf{k}} + v_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}} \} \times \\ & \times \left[\frac{v_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}}}{\mathcal{E}_K + E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}}} + \frac{u_{\mathbf{K}/2+\mathbf{k}} u_{\mathbf{K}/2-\mathbf{k}}}{-\mathcal{E}_K + E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}}} \right] = 1 \end{aligned} \quad (31)$$

where φ is the angle between \mathbf{K} and \mathbf{k} . Here $k_D \equiv \omega_D/v_F$; note difference with definition just below (15). As before $\lambda \equiv VN(E_F)$ with $N(E_F) \equiv m/2\pi\hbar^2$ the constant 2D electronic DOS and $V > 0$ is the interaction strength defined in (20). Angle-resolved photoemission spectral studies of *BiSrCaCuO* have shown evidence [69] in this cuprate for the Bogoliubov functions u_k^2 and v_k^2 , both above and below the Fermi energy.

The *ABH collective excitation mode* energy \mathcal{E}_K must then be extracted from this equation. For $\mathbf{K} = 0$ it is just $\mathcal{E}_0 = 0$ (Ref. [9] p. 39). Then (31) rewritten as an integral over $\xi \equiv \hbar^2 k^2/2m - E_F$ reduces to

$$\int_0^{\hbar\omega_D} d\xi / \sqrt{\xi^2 + \Delta^2} = 1/\lambda, \quad (32)$$

or the familiar BCS $T = 0$ gap equation for interaction (20). The integral is exact and gives

$$\Delta = \hbar\omega_D / \sinh(1/\lambda). \quad (33)$$

Returning to the ABH energy \mathcal{E}_K equation (31) and Taylor-expanding \mathcal{E}_K about $K = 0$ and small λ leaves

$$\mathcal{E}_K = \frac{\hbar v_F}{\sqrt{2}} K + O(K^2) + o(\lambda), \quad (34)$$

where $o(\lambda)$ denote interfermion interaction correction terms that vanish as $\lambda \rightarrow 0$. Note that the leading term is just the ordinary sound mode in an IFG whose sound speed $c = v_F/\sqrt{d}$ in d dimensions.

The latter also follows elementarily on solving for c in the familiar thermodynamic relation $dP/dn = mc^2$ involving the zero-temperature IFG pressure

$$\begin{aligned} P &= n^2[d(E/N)/dn] = 2nE_F/(d+2) \\ &= 2C_d n^{2/d+1}/(d+2) \end{aligned} \quad (35)$$

where the constant C_d will drop out. Here the IFG ground-state (internal) energy per fermion $E/N = dE_F/(d+2) = C_d n^{2/d}$ was used along with $E_F \equiv \hbar^2 k_F^2/2m$ and

$$n \equiv N/L^d = k_F^d/d2^{d-2}\pi^{d/2}\Gamma(d/2) \quad (36)$$

for the fermion-number density n . The derivative of (35) with respect to n finally gives $c = \hbar k_F/m\sqrt{d} \equiv v_F/\sqrt{d}$, which in 2D is just the leading term in (34).

5.2 Moving CP (nontrivial) solution

The nontrivial *moving CP* solution of the BCS-ground-state-based BS treatment, which is *entirely new*, comes from the remaining 1×1 determinant. It leads to the pair energy \mathcal{E}_K which in 2D is contained in the equation

$$\begin{aligned} &\frac{1}{2\pi}\lambda\hbar v_F \int_{k_F-k_D}^{k_F+k_D} dk \int_0^{2\pi} d\varphi u_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}} \times \\ &\times \{u_{\mathbf{K}/2-\mathbf{k}} v_{\mathbf{K}/2+\mathbf{k}} - u_{\mathbf{K}/2+\mathbf{k}} v_{\mathbf{K}/2-\mathbf{k}}\} \frac{E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}}}{-\mathcal{E}_K^2 + (E_{\mathbf{K}/2+\mathbf{k}} + E_{\mathbf{K}/2-\mathbf{k}})^2} = 1. \end{aligned} \quad (37)$$

In addition to the pp and hh wavefunctions (depicted diagrammatically in Ref. [54] Fig. 2), diagrams associated with the ph channel give zero contribution at $T = 0$. A third equation for the ph wavefunction describes the ph bound state but turns out to depend only on the pp and hh wavefunctions. Taylor-expanding \mathcal{E}_K in (37) in powers of K around $K = 0$, and introducing a possible damping factor by adding an imaginary term $-i\Gamma_K$ in the denominator, yields to order K^2

$$\begin{aligned} \pm\mathcal{E}_K &\simeq 2\Delta + \frac{\lambda}{2\pi}\hbar v_F K + \frac{1}{9}\frac{\hbar v_F}{k_D}e^{1/\lambda}K^2 \\ &- i\left[\frac{\lambda}{\pi}\hbar v_F K + \frac{1}{12}\frac{\hbar v_F}{k_D}e^{1/\lambda}K^2\right] + O(K^3) \end{aligned} \quad (38)$$

where the upper and lower signs refer to 2p- and 2h-CPs, respectively. A linear dispersion in leading order again appears, but now associated with the bosonic moving CP. Hence the *positive*-energy 2p-CP resonance has a width Γ_K and a lifetime

$$\tau_K \equiv \hbar/2\Gamma_K = \hbar/2\left[(\lambda/\pi)\hbar v_F K + (\hbar v_F/12k_D)e^{1/\lambda}K^2\right]. \quad (39)$$

This diverges only at $K = 0$, falling to zero as K increases. Thus, “faster” moving CPs are shorter-lived and eventually break up, while “non-moving” ones are infinite-lifetime stationary states. The linear term $(\lambda/2\pi)\hbar v_F K$ in (38) contrasts sharply with the *coupling-independent* leading-term in (18) [or $1/2$ in 3D in (19), Ref. [34] p. 33, instead of $2/\pi$] that follows from the *original* CP problem (25) neglecting holes. This holds for either interaction (20) [43] *or* for an attractive delta interfermion potential well in 2D [40] or in 3D [41]. These δ -wells are imagined regularized [39] to possess a single bound level whose binding energy (in 2D) or scattering length (in 3D) serve as the coupling parameter. Figure 4a shows the exact moving CP (mCP) energy

(full curves) extracted from (37), along with its leading linear-dispersion term (thin short-dashed lines) and this plus the next (quadratic) term (long-dashed curves) from (38). The interaction parameter values used with (20) were $\hbar\omega_D/E_F = 0.05$ (a typical value for cuprates) and the two values $\lambda = \frac{1}{4}$ and $\frac{1}{2}$. Using (33) in (38) gives

$$\mathcal{E}_0/E_F \equiv 2\Delta/E_F = 2\hbar\omega_D/E_F \sinh(1/\lambda), \quad (40)$$

having the values $\simeq 0.004$ and 0.028 , respectively (marked as dots on the figure ordinate). Remarkably enough, the linear approximation (thin short-dashed lines in figure) is better over a wider range of K/k_F values for weaker coupling (lower set of three curves) in spite of a larger and larger (because of the factor $e^{1/\lambda}$) partial contribution from the quadratic term in (38). This peculiarity also emerged from the ordinary CP treatment of Sec. 3, Refs. [40][41][43]. It suggests the expansion in powers of K to be an asymptotic series that should be truncated after the linear term. For reference we also plot the linear leading term $\hbar v_F K/\sqrt{2}$ of the sound solution (34). We note that the *coupling-independent* leading-term [43] $(2/\pi)\hbar v_F K$ from the *original* CP problem neglecting holes, if graphed in Fig. 4, would almost coincide with the ABH term $\hbar v_F K/\sqrt{2}$ and have a slope about 90% smaller.

Empirical evidence for the *linearly-dispersive* nature of Cooper pairs in cuprates has been argued by Wilson [70] to be suggested by the scanning tunneling microscope conductance scattering data in BSCCO obtained by Davis and coworkers [71][72]. More suggestive direct evidence is shown in Figure 5 [73] with experimental data (mostly from penetration-depth measurements) for two 3D SCs [74][75], two quasi-2D cuprates [76]-[78], and a quasi-1D SC [79]. The data are seen to agree quite well, at least for $T \gtrsim 0.5T_c$, with the *pure-phase* (only 2e- or 2h-CP) BEC condensate fraction formula $1 - (T/T_c)^{d/s}$ for $d = 3, 2$ and 1 , respectively, *provided one assumes* $s = 1$. For lower T 's, one can argue on the basis of Fig. 9 below that a *mixed* BEC phase containing both 2e- and 2h-CPs becomes more stable (i.e., has lower Helmholtz free energy) so that the simple pure-phase $1 - (T/T_c)^{d/s}$ formula is no longer strictly valid.

As in Cooper's [29] original equation (25), the BS moving CPs are characterized by a definite \mathbf{K} and *not* also by definite \mathbf{k} as the pairs discussed by BCS [7]. Hence, the objection does not apply that CPs are not bosons because BCS pairs with definite \mathbf{K} and \mathbf{k} (or equivalently definite \mathbf{k}_1 and \mathbf{k}_2) have creation/annihilation operators that do *not* obey the usual Bose commutation relations [Ref. [7], Eqs. (2.11) to (2.13)]. In fact, either (25) or (37) shows that a given "ordinary" or "generalized" CP state labeled by either \mathbf{K} or \mathcal{E}_K can accommodate (in the thermodynamic limit) an indefinitely many possible BCS pairs with different \mathbf{k} 's; see Ref. [82] or Appendix A. A recent electronic analog [83] of the Hanbury Brown-Twiss photon-effect experiment suggest electron pairs in a SC to be definitely bosons.

To conclude this section, hole pairs treated on a par with electron pairs play a vital role in determining the precise nature of CPs even at zero temperature—only when based not on the usual IFG "sea" but on the BCS ground state. Their treatment with a Bethe-Salpeter equation gives purely-imaginary-energy CPs when based on the IFG, and positive-energy resonant-state CPs with a finite lifetime for nonzero CMM when based on the BCS ground state—instead of the more familiar negative-energy stationary states of the original IFG-based CP problem that neglects holes, as sketched just below (24). The BS "moving-CP" dispersion relation (38), on the other hand, resembles the plasmon dispersion curve in 3D. It is gapped by twice the BCS energy gap, followed by a *linear* leading term in the CMM expansion about $K = 0$, instead of the quadratic for the 3D plasmon curve. This linearity is distinct from the better-known one (34) associated with the sound or ABH collective excitation mode whose energy vanishes at $K = 0$. Thus, BF models assuming this CP linearity for the boson component, instead of the quadratic $\hbar^2 K^2/2(2m)$ assumed in Refs. [3], [11]-[14], [89]-[92] among many others, can give BEC for all $d > 1$, including exactly 2D. Such BF models can then in principle address not only quasi-2D cuprate but also quasi-1D organo-metallic superconductors.

6 The CBFM Hamiltonian

The CBFM [13, 14] is described (in d dimensions) by the Hamiltonian $H = H_0 + H_{int}$. The unperturbed Hamiltonian H_0 corresponds to a “normal” state which is an *ideal* (i.e., noninteracting) gas mixture of unpaired fermions and both types of CPs, two-electron (2e) and two-hole (2h), and is given by

$$H_0 = \sum_{\mathbf{k}_1, s_1} \epsilon_{\mathbf{k}_1} a_{\mathbf{k}_1, s_1}^+ a_{\mathbf{k}_1, s_1} + \sum_{\mathbf{K}} E_+(K) b_{\mathbf{K}}^+ b_{\mathbf{K}} - \sum_{\mathbf{K}} E_-(K) c_{\mathbf{K}}^+ c_{\mathbf{K}} \quad (41)$$

where as before $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2$ is the CP CMM wavevector, $\mathbf{k} \equiv \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$ its relative wavevector, while $\epsilon_{\mathbf{k}_1} \equiv \hbar^2 k_1^2 / 2m$ are the single-electron, and $E_{\pm}(K)$ the 2e-/2h-CP *phenomenological*, energies. Here $a_{\mathbf{k}_1, s_1}^+$ ($a_{\mathbf{k}_1, s_1}$) are creation (annihilation) operators for fermions and similarly $b_{\mathbf{K}}^+$ ($b_{\mathbf{K}}$) and $c_{\mathbf{K}}^+$ ($c_{\mathbf{K}}$) for 2e- and 2h-CP bosons, respectively. Two-hole CPs are considered *distinct* and *kinematically independent* from 2e-CPs since their Bose commutation relations involve a relative sign change, in sharp contrast with electron or hole fermions whose Fermi anticommutation relations do not. Our present formulation is of course nonrelativistic because of relatively low temperatures.

At the opposite extreme of very high T 's (compared with the boson rest-mass energy), however, one has a relativistic regime where pair production takes place and BEC *must* take \tilde{N} antibosons of charge, say, $-q$ into account along with the N bosons of charge q . In units such that $\hbar \equiv c \equiv k_B \equiv 1$ the boson energy is $\varepsilon_K = (K^2 + m_B^2)^{1/2}$. Charge conservation requires that not $N \equiv N_0(T) + \sum_{\mathbf{K} \neq 0} [\exp \beta(\varepsilon_K - \mu_B) - 1]^{-1}$ be constant but rather $N - \tilde{N}$, where \tilde{N} is the same expression as N but with $+\mu_B$ instead of $-\mu_B$. If $\rho \equiv q(N - \tilde{N})/L^3 \equiv qn$ is the net conserved charge density, it is shown in Ref. [93] that $T_c = (3|n|/m_B)^{1/2}$ and that the condensate fraction $n_0(T)/n = 1 - (T/T_c)^2$. This is *qualitatively* different from the better-known results assuming only N constant, which are the mass-independent $T_c = [\pi^2 n / \zeta(3)]^{1/3}$, and $n_0(T)/n = 1 - (T/T_c)^3$. This analogy with the CBFM exhibits the strikingly dramatic effect of including or not antiparticles—or, in our nonrelativistic problem, of including or not hole pairs.

The interaction Hamiltonian H_{int} consists of four distinct BF interaction vertices, see Fig. 6, each with two-fermion/one-boson creation or annihilation operators, depicting how unpaired electrons (subindex +) [or holes (subindex -)] combine to form the 2e- (and 2h-CPs) assumed in the d -dimensional system of size L , namely

$$\begin{aligned} H_{int} = & L^{-d/2} \sum_{\mathbf{k}, \mathbf{K}} f_+(k) \{ a_{\mathbf{k}+\frac{1}{2}\mathbf{K}, \uparrow}^+ a_{-\mathbf{k}+\frac{1}{2}\mathbf{K}, \downarrow}^+ b_{\mathbf{K}} + a_{-\mathbf{k}+\frac{1}{2}\mathbf{K}, \downarrow} a_{\mathbf{k}+\frac{1}{2}\mathbf{K}, \uparrow} b_{\mathbf{K}}^+ \} \\ & + L^{-d/2} \sum_{\mathbf{k}, \mathbf{K}} f_-(k) \{ a_{\mathbf{k}+\frac{1}{2}\mathbf{K}, \uparrow}^+ a_{-\mathbf{k}+\frac{1}{2}\mathbf{K}, \downarrow}^+ c_{\mathbf{K}}^+ + a_{-\mathbf{k}+\frac{1}{2}\mathbf{K}, \downarrow} a_{\mathbf{k}+\frac{1}{2}\mathbf{K}, \uparrow} c_{\mathbf{K}} \}. \end{aligned} \quad (42)$$

Note that the *fermion-pair interaction* H_{int} is reminiscent of the Fröhlich (or Dirac QED) interaction Hamiltonian (Ref. [44], p. 396 ff.) involving two fermion and one boson operators, but with *two* types of CPs instead of phonons (or photons). But in contrast with Fröhlich and Dirac there is no a conservation law for the number of unpaired electrons, i.e., $[H_{int}, \sum_{\mathbf{k}_1, s_1} \epsilon_{\mathbf{k}_1} a_{\mathbf{k}_1, s_1}^+ a_{\mathbf{k}_1, s_1}] \neq 0$. (Note however that $[H_{int}, \sum_{\mathbf{k}_1, s_1} \mathbf{k}_1 a_{\mathbf{k}_1, s_1}^+ a_{\mathbf{k}_1, s_1}] = 0$ and $[H_{int}, \sum_{\mathbf{k}_1, s_1} s_1 a_{\mathbf{k}_1, s_1}^+ a_{\mathbf{k}_1, s_1}] = 0$.) Just as the Fröhlich (or Dirac) interaction Hamiltonians are the most natural ones to use in a many-electron/phonon (or photon) system, one can conjecture the same of (42) for the BF system under study. Indeed, this H_{int} has *formally* already been employed under various guises by several authors [10]-[12],[84]-[87], but *without* hole pairs. More recently, a similar BF H_{int} has been employed [89]-[92] to study quantum degenerate Fermi gases consisting of neutral ^{40}K atoms and their so-called Feshbach “resonance superfluidity.” However, these authors assume quadratically-dispersive CPs, besides also *excluding* 2h-CPs and cannot thus relate their formalism to BCS theory.

The energy form factors $f_{\pm}(k)$ in (42) are essentially the Fourier transforms of the 2e- and 2h-CP intrinsic wavefunctions, respectively, in the relative coordinate between the paired fermions of the CP. In Refs.[13][14] they are taken simply as

$$f_{+}(\epsilon) = \begin{cases} f & \text{for } E_f < \epsilon < E_f + \delta\epsilon, \\ 0 & \text{otherwise,} \end{cases} \quad (43)$$

$$f_{-}(\epsilon) = \begin{cases} f & \text{for } E_f - \delta\epsilon < \epsilon < E_f, \\ 0 & \text{otherwise.} \end{cases} \quad (44)$$

The quantities E_f and $\delta\epsilon$ are *new* phenomenological dynamical energy parameters (in addition to the positive BF vertex coupling parameter f) that replace the previous such $E_{\pm}(0)$, through the relations

$$E_f \equiv \frac{1}{4}[E_{+}(0) + E_{-}(0)] \quad \text{and} \quad \delta\epsilon \equiv \frac{1}{2}[E_{+}(0) - E_{-}(0)], \quad (45)$$

where $E_{\pm}(0)$ are the (empirically *unknown*) zero-CMM energies of the 2e- and 2h-CPs, respectively. Clearly $E_{\pm}(0) = 2E_f \pm \delta\epsilon$. The quantity E_f will serve as a convenient energy scale and is not to be confused with the Fermi energy $E_F = \frac{1}{2}mv_F^2 \equiv k_B T_F$ where T_F is the Fermi temperature. The Fermi energy E_F equals $\pi\hbar^2 n/m$ in 2D and $(\hbar^2/2m)(3\pi^2 n)^{2/3}$ in 3D, with n the total number-density of charge-carrier electrons. The quantities E_f and E_F coincide *only* when perfect 2e/2h-CP symmetry holds.

The interaction Hamiltonian (42) can be further simplified by keeping only the $\mathbf{K} = 0$ terms. One can then apply the Bogoliubov “recipe” [68] (see, also e.g., [44] p. 199) of replacing in the full Hamiltonian $H = H_0 + H_{int}$ all zero-CMM 2e- and 2h-CP boson creation and annihilation operators by their respective c-numbers $\sqrt{N_0}$ and $\sqrt{M_0}$, where $N_0(T)$ and $M_0(T)$ are the number of zero-CMM 2e- and 2h-CPs, respectively. One eventually seeks the lowest critical temperature T_c such that, e.g., $N_0(T_c)$ or $M_0(T_c)$ vanish. Note that T_c calculated thusly can, in principle, turn out to be zero, in which case there is no BEC. Now, $H - \mu\hat{N}$ can be diagonalized exactly via a Bogoliubov-Valatin transformation [45] in terms of the thermodynamic potential $\Omega \equiv -PL^d$ for the CBFM, with P its pressure and L^d the system “volume,” which is

$$\Omega(T, L^d, \mu, N_0, M_0) = -k_B T \ln \left[\text{Tr} e^{-\beta(H - \mu\hat{N})} \right], \quad (46)$$

where “Tr” stands for “trace.” Inserting (41) and (42) into (46) one obtains [13] after some algebra

$$\begin{aligned} & \Omega(T, L^d, \mu, N_0, M_0)/L^d = \\ & = \int_0^\infty d\epsilon N(\epsilon) [\epsilon - \mu - E(\epsilon)] - 2k_B T \int_0^\infty d\epsilon N(\epsilon) \ln\{1 + \exp[-\beta E(\epsilon)]\} + \\ & + [E_{+}(0) - 2\mu] n_0 + k_B T \int_{0+}^\infty d\eta M(\eta) \ln\{1 - \exp[-\beta \mathcal{E}_{+}(\eta)]\} + \\ & + [2\mu - E_{-}(0)] m_0 + k_B T \int_{0+}^\infty d\eta M(\eta) \ln\{1 - \exp[-\beta \mathcal{E}_{-}(\eta)]\}. \end{aligned} \quad (47)$$

For $d = 3$ one has

$$N(\epsilon) \equiv \frac{m^{3/2}}{2^{1/2}\pi^2\hbar^3} \sqrt{\epsilon} \quad \text{and} \quad M(\eta) \equiv \frac{2m^{3/2}}{\pi^2\hbar^3} \sqrt{\eta} \quad (48)$$

for the (one-spin) fermion and boson DOS at energies $\epsilon = \hbar^2 k^2/2m$ and $\eta = \hbar^2 K^2/2(2m)$, respectively. The latter is an *assumption* to be lifted later so as to include Fermi sea effects which transform the boson dispersion relation from quadratic to linear, as already discussed in Sections 3 to 5. Also, recalling the 2e- and 2h-boson energies $E_{\pm}(K)$ introduced in (41) the (bosonic) energies $\mathcal{E}_{+}(\eta)$ and $\mathcal{E}_{-}(\eta)$ to simplify notation have been defined as

$$E_{+}(K) - 2\mu = E_{+}(0) - 2\mu + \eta \equiv \mathcal{E}_{+}(\eta), \quad (49)$$

$$2\mu - E_{-}(K) = 2\mu - E_{-}(0) + \eta \equiv \mathcal{E}_{-}(\eta). \quad (50)$$

Finally, the relation between the fermion spectrum $E(\epsilon)$ and fermion energy gap $\Delta(\epsilon)$ is of the form

$$E(\epsilon) = \sqrt{(\epsilon - \mu)^2 + \Delta^2(\epsilon)}, \quad (51)$$

$$\Delta(\epsilon) \equiv \sqrt{n_0}f_+(\epsilon) + \sqrt{m_0}f_-(\epsilon). \quad (52)$$

This last expression for the gap $\Delta(\epsilon)$ implies a simple T -dependence rooted in the 2e-CP $n_0(T) \equiv N_0(T)/L^d$ and 2h-CP $m_0(T) \equiv M_0(T)/L^d$ number densities of BE-condensed bosons, i.e.,

$$\Delta(T) = \sqrt{n_0(T)}f_+(\epsilon) + \sqrt{m_0(T)}f_-(\epsilon). \quad (53)$$

A $\Delta(T)^2$ temperature dependence in the order parameter in thin films of the $Tl_2Ba_2CaCu_2O_8$ cuprate suggests itself from thermal difference reflectance spectra [95]. This implies that condensate densities $n_0(T)$ or $m_0(T)$ might be more fundamental as order parameters than $\Delta(T)$, at least for this material.

Minimizing $\Omega(T, L^d, \mu, N_0, M_0)$ with respect to N_0 and M_0 , and simultaneously fixing the total number N of electrons by introducing the electron chemical potential μ , one thus specifies an *equilibrium state* of system with volume L^d and temperature T by requiring that

$$\frac{\partial \Omega}{\partial N_0} = 0, \quad \frac{\partial \Omega}{\partial M_0} = 0, \quad \text{and} \quad \frac{\partial \Omega}{\partial \mu} = -N. \quad (54)$$

Here N evidently includes both paired and unpaired CP fermions. Some algebra then leads to the three coupled transcendental Eqs. (7)-(9) of Ref. [13]. These can be rewritten somewhat more transparently as: a) two “*gap-like equations*”

$$2\sqrt{n_0}[E_+(0) - 2\mu] = \int_0^\infty d\epsilon N(\epsilon) \frac{\Delta(\epsilon)f_+(\epsilon)}{E(\epsilon)} \tanh \frac{1}{2}\beta E(\epsilon); \quad (55)$$

$$2\sqrt{m_0}[2\mu - E_-(0)] = \int_0^\infty d\epsilon N(\epsilon) \frac{\Delta(\epsilon)f_-(\epsilon)}{E(\epsilon)} \tanh \frac{1}{2}\beta E(\epsilon), \quad (56)$$

and b) a single “*number equation*” (that ensures charge conservation)

$$2n_B(T) - 2m_B(T) + n_f(T) = n. \quad (57)$$

Here $n \equiv N/L^3$ is the number density of electrons while $n_B(T)$ and $m_B(T)$ are the number densities of 2e- and 2h-CPs in *all* bosonic states, respectively, and are given by

$$n_B(T) \equiv n_0(T) + \int_{0+}^\infty d\eta M(\eta) \frac{1}{e^{\beta \mathcal{E}_+(\eta)} - 1}, \quad (58)$$

$$m_B(T) \equiv m_0(T) + \int_{0+}^\infty d\eta M(\eta) \frac{1}{e^{\beta \mathcal{E}_-(\eta)} - 1}. \quad (59)$$

Clearly,

$$n_f(T) \equiv \int_0^\infty d\epsilon N(\epsilon) \left[1 - \frac{\epsilon - \mu}{E(\epsilon)} \tanh \frac{1}{2}\beta E(\epsilon) \right] \quad (60)$$

is the number density of unpaired electrons at any T . Self-consistent (at worst, numerical) solution of the *three coupled equations* (55) to (57) yields the three thermodynamic variables of the CBFM

$$n_0(T, n, \mu), \quad m_0(T, n, \mu), \quad \text{and} \quad \mu(T, n). \quad (61)$$

The pressure P , entropy S and specific heat at constant volume C of an equilibrium state characterized by T and n are then given by

$$P(T, n) = -\Omega/L^d, \quad S(T, n)/L^d = -k_B \frac{\partial}{\partial T} (\Omega/L^d), \quad (62)$$

$$C(T, n)/L^d = T \frac{\partial}{\partial T} [S(T, n)/L^d], \quad (63)$$

all evaluated at fixed $n_0(T, \mu, n)$, $m_0(T, \mu, n)$ and $\mu(T, n)$. The Helmholtz free energy

$$F(T, L^d, N) \equiv E - TS \equiv \Omega + \mu N \quad (64)$$

where E is the internal energy, then follows from

$$F(T, n)/L^d = -P(T, n) + n \mu(T, n). \quad (65)$$

Finally, the critical magnetic field is

$$\begin{aligned} H_c^2(T, n)/8\pi &\equiv F_n(T, n)/L^d - F_s(T, n)/L^d = \\ &= P_s(T, n) - P_n(T, n) + [\mu_n(T, n) - \mu_s(T, n)] n, \end{aligned} \quad (66)$$

with subscripts s and n meaning “superconducting” and “normal” states.

7 Main statistical theories as special cases of CBFM

Most significantly, the three CBFM equations just stated contain *five* different theories as special cases, see flow chart in Fig. 7. Perfect $2e/2h$ CP symmetry $n_B(T) = m_B(T)$ and $n_0(T) = m_0(T)$ can be seen from (55) and (56), as well as from $E_{\pm}(0) = 2E_f \pm \delta\varepsilon$, to imply that E_f coincides with μ . The CBFM then reduces to:

i) the gap and number equations of the *BCS-Bose crossover picture* [96] for the BCS model interaction—if the BCS parameters V and Debye energy $\hbar\omega_D$ are identified with the BF interaction Hamiltonian H_{int} parameters $f^2/2\delta\varepsilon$ and $\delta\varepsilon$, respectively. The crossover picture for unknowns $\Delta(T)$ and $\mu(T)$ is now supplemented by the key relation

$$\Delta(T) = f \sqrt{n_0(T)} = f \sqrt{m_0(T)}. \quad (67)$$

The 35-year-old crossover picture is associated with names such as (in chronological order) Friedel and co-workers [97], Eagles [98], Leggett [99], Miyake [100], Noziers [101], Micnas, Ranninger & Robaszkiewicz [10], Randeria [102], van der Marel [103], Bar-Yam [85], Drechsler & Zwerger [104], Hausmann [105], Pistoletti & Strinati [106], and many others. However, it seems to be a very modest improvement over BCS theory since the unphysically large λ of about 8 is required to bring $\mu(T_c)/E_F$ down from 1.00 to 0.998 [107]. If one imposes that $\mu(T_c) = E_F$ exactly, as follows for weak BF coupling f from the number equation, the crossover picture is well-known to reduce to:

ii) *ordinary BCS theory* which is characterized by a *single* equation, the gap equation for all T . Thus, *the BCS condensate is precisely a BE condensate* whenever both $n_B(T) = m_B(T)$ and $n_0(T) = m_0(T)$ and the BF coupling f is small. Indeed, for small coupling λ the CBFM $T = 0$ superconducting state has a *lower* energy than the corresponding BCS state, see Appendix B. The BCS state comes from a variational trial wave function so that its energy expectation value by the Rayleigh-Ritz variational principle is a rigorous upper bound to the exact value. Thus, the CBFM has a somewhat larger condensation energy than that of BCS theory, but both *coincide* in leading order. In addition, at least two universal constants, the gap-to- T_c ratio $2\Delta(0)/k_B T_c$ and the

specific heat jump $\Delta C_s(T_c)$ to the normal value $C_n(T_c)$, have been shown [14] to coincide with the BCS values of 3.53 and 1.43 in this limit of the CBFM.

On the other hand, for no 2h-CPs present the CBFM reduces [13] also to:

iii) the *BEC BF model* in 3D of Friedberg and Lee [11, 12] characterized by the relation $\Delta(T) = f\sqrt{n_0(T)}$. With just *one* adjustable parameter (the ratio of perpendicular to planar boson masses) this theory fitted [12] cuprate T_c/T_F data quite well. The ratio turned out to be 66,560—just under the 10^5 anisotropy ratio reported [108] almost contemporaneously for *BSCO*. Finally, when $f = 0$ this model reduces to:

iv) the ideal BF model (IBFM) of Ref. [17, 18] that predicts nonzero BEC T_{cs} even in 2D. The “gapless” IBFM cannot describe the superconducting phase. But considered as a model for the *normal state* it should provide feasible T_{cs} as singularities within a BE scenario that are approached from *above* T_c . Figure 8 displays the T_c predictions [18] in 2D for cuprate superconductors with *no* adjustable parameters. Finally, the CBFM reduces to:

v) the familiar T_c -formula of ordinary BEC in 3D, albeit as an *implicit* equation with the boson number-density being T -dependent.

8 Enhanced T_c s from the CBFM

The very general CBFM has been applied and gives sizeable enhancements in T_{cs} over BCS theory that emerge for moderate departures from perfect 2e/2h-pair symmetry. This is attained for the *same* interaction model (with the same coupling strength λ and cutoff $\hbar\omega_D$ parameters) used in conventional superconductors. The BF (two-fermion) interaction (42) with (43) and (44) bears a one-to-one correspondence with the more familiar “direct” four-fermion electron-phonon interaction, mimicked, e.g., by the BCS model interaction. Its double Fourier transform is a negative constant $-V$, nonzero only within an energy shell $2\hbar\omega_D$ about the Fermi surface, with ω_D the Debye frequency. The correspondence is realized [13, 14] by setting $f^2/2\delta\varepsilon \equiv V$ and $\delta\varepsilon \equiv \hbar\omega_D$, from which the familiar dimensionless BCS model interaction parameters $\lambda \equiv N(E_F)V$ and $\hbar\omega_D/E_F$ are then recovered.

The three coupled equations (55) to (57) of the CBFM that determine the d -dimensional BE-condensate number-densities $n_0(T)$ and $m_0(T)$ of 2e- and 2h-CPs, respectively, as well as the fermion chemical potential $\mu(T)$, were solved numerically in 3D for $\lambda = 1/5$ and $\hbar\omega_D/E_F = 0.001$ in Ref. [14] assuming a quadratic boson dispersion relation $\eta = \hbar^2 K^2/2(2m)$, i.e., $s = 2$ in (1). For this case Figure 9 maps the phase diagram in the vicinity of the BCS T_c value (marked BCS-B in the figure) at $\Delta n \equiv n/n_f - 1 = 0$ (corresponding to perfect 2e/2h-CP symmetry) where n_f is a very special case of $n_f(T)$ as defined in (60). Namely, n_f can be seen [14] to be the number of unpaired electrons at zero coupling and temperature. Besides the *normal* phase (n) consisting of the ideal BF gas described by H_0 , three different stable BEC phases emerged—plus several metastable, i.e., of higher Helmholtz free energy. These are two *pure* phases of *either* 2e-CP ($s+$) or 2h-CP ($s-$) BE-condensates, and a lower temperature *mixed* phase (ss) with arbitrary proportions of 2e- and 2h-CPs, see Fig. 9. Of greater physical interest are the two higher- T_c *pure* phases so that we focus below only on them. For each pure phase at a critical temperature we have *either* $\Delta(T_{cs+}) = f\sqrt{n_0(T_{cs+})} \equiv 0$ or $\Delta(T_{cs-}) = f\sqrt{m_0(T_{cs-})} \equiv 0$, where $\Delta(T)$ is the electronic (BCS-like) energy gap. Their intersection gives the BCS T_c value of $7.64 \times 10^{-6}T_F$ that also follows from the familiar BCS expression

$$T_c/T_F \simeq 1.134(\hbar\omega_D/E_F) \exp(-1/\lambda). \quad (68)$$

We next focus on $s = 1$ which, as we saw in Sections 3 to 5, occurs in the leading term for “ordinary” CPs in a Fermi sea as well as for “generalized” CPs in a BCS state. For these, the boson energy η in (47) to be used

later has leading terms in the many-body Bethe-Salpeter (BS) CP dispersion relation that are

$$\eta \simeq (\lambda/2\pi)\hbar v_F K \quad \text{2D} \quad (69)$$

[53] and

$$\eta \simeq (\lambda/4)\hbar v_F K \quad \text{3D} \quad (70)$$

[54]. As before, $\lambda \equiv VN(E_F)$ where $N(E_F)$ is the electron DOS (for one spin) at the Fermi surface. Note that the boson energy η is *not* the quadratic $\hbar^2 K^2/2(2m)$ appropriate for a composite boson of mass $2m$ moving not in the Fermi sea but in vacuum [3]-[6], [11]-[14], and assumed in (48). Note also that the BS CP linear dispersion coefficient $\lambda/2\pi$ in 2D (or $\lambda/4$ in 3D) contrasts markedly with the coupling-independent $2/\pi$ coefficient in 2D (or $1/2$ in 3D, as apparently first quoted in Ref. [34], p. 33) obtained [43] in the *simple* CP problem [29] which ignores holes. Thus, again with $n_B = n/2$, for $s = 1$ and $C_1 = \lambda b(d)\hbar v_F$ with $b(2) = 1/2\pi$ and $b(3) = 1/4$ according to (69) and (70), (C.1) leads to

$$T_c/T_F = 2\lambda b(d)/[d\Gamma(d)\zeta(d)]^{1/d}. \quad (71)$$

For $\lambda = 1/2$ this is $\simeq 0.088$ if $d = 2$ since $\zeta(2) = \pi^2/6$, and $\simeq 0.129$ if $d = 3$ since $\zeta(3) \simeq 1.202$. These two values will appear as the black squares in Figs. 10 to 13 (upper ones in Figs. 10 and 11). They mark the BEC limiting values if *all* electrons in our 2D or 3D many-electron system were imagined paired into noninteracting bosons formed with the BCS model interelectron interaction. For $\lambda = 1/4$ the lower black squares in Figs. 10 and 11 apply.

8.1 Two dimensions (2D)

To address cuprates and other copper-plane-free planar superconductors (such as $Sr_2YRu_{1-x}Cu_xO_6$ with $T_c \simeq 49$ K, which is *Cu*-doped but has no *CuO* planes) we deal first with the CBFM in 2D. Although it is still controversial [109] as to *which* planes, the *CuO* or *BaO* or *SrO*, etc., the superconductivity resides in, these planes are parallel to each other, say, in the *ab* directions. Resistivity ρ anisotropies ρ_c/ρ_{ab} , with the *c* direction perpendicular to the *ab* denoting the *CuO* or *BaO* or *SrO* planes, can be as high as 10^5 in $Bi_{2+x}Sr_{2-y}CuO_{6+\delta}$ [108], if not higher, even though “only” about 10^2 in $YBa_2Cu_3O_{7-\delta}$ (*YBCO*). From the Drude 1900 resistivity model [110] $\rho = m/ne^2\tau$ for current carriers of charge e , effective mass m , number density n , and τ is some average time between collisions. Thus, if $\rho_c/\rho_{ab} = m_c/m_{ab}$ is ∞ one has a precisely 2D situation; if it is 1 we have the perfectly isotropic 3D case. Hence, the large (10^5) but finite ratio observed implies $(2 + \epsilon)D$ or “quasi-2D.” For cuprates the value $d \simeq 2.03$ has been extracted independently by two groups [111][112] for *YBCO* as more realistic than $d = 2$, since that value reflects inter-CuO (or BaO)-layer couplings. Our results in that case would be very close to those for $d = 2$ since, e.g., from Appendix C (C.1), T_c for $s = 1$ (but, perhaps very significantly, *not* for $s = 2$) varies little [113] with d around $d = 2$. This justifies models in exactly 2D, at least as a very good initial approximation.

In 2D the electronic DOS per unit area L^2 is constant, namely $N(\varepsilon) = m/2\pi\hbar^2$. Using the leading term of the BS CP *linear* dispersion relation (69) we get for the bosonic DOS

$$M(\eta) \equiv (1/2\pi)K(dK/d\eta) \simeq (2\pi/\lambda^2\hbar^2v_F^2)\eta \quad (72)$$

instead of the constant that follows in 2D from the quadratic dispersion $\eta = \hbar^2 K^2/2(2m)$. Employing $E_f \equiv \pi\hbar^2 n_f/m = k_B T_f$ as energy/density/temperature scaling factors, and the relation $n/n_f = (E_f/E_f)^{d/2}$ to convert quantities such as T_c/T_f to T_c/T_F , where $E_F \equiv k_B T_F$, the two “working equations” for the *pure 2e-CP*

phase [i.e., $m_B(T_c) \equiv 0$] (with all quantities dimensionless, energies in units of E_f and electron particle-densities in units of n_f) become

$$1 + \hbar\omega_D/2 - \mu = \lambda(\hbar\omega_D/2) \int_1^{1+\hbar\omega_D} dx \frac{1}{|x-\mu|} \tanh \frac{|x-\mu|}{2T_c}, \quad (73)$$

$$\frac{1}{2} \int_0^\infty dx [1 - \tanh \frac{x-\mu}{2T_c}] + \frac{\pi^2}{\lambda^2} \frac{1}{n} \int_0^\infty dx x [\coth \frac{x+2(1+\hbar\omega_D/2-\mu)}{2T_c} - 1] = n. \quad (74)$$

These are just the gap-like equation associated with 2e-CPs and its corresponding number equation. Recall that n_f can be seen [13] to be the number-density of unpaired electrons at zero T and zero λ .

For the *pure 2h-CP phase* (i.e., $n_B \equiv 0$) the two working equations are

$$\mu - 1 + \hbar\omega_D/2 = \lambda(\hbar\omega_D/2) \int_{1-\hbar\omega_D}^1 dx \frac{1}{|x-\mu|} \tanh \frac{|x-\mu|}{2T_c}, \quad (75)$$

$$\frac{1}{2} \int_0^\infty dx [1 - \tanh \frac{x-\mu}{2T_c}] - \frac{\pi^2}{\lambda^2} \frac{1}{n} \int_0^\infty dx x [\coth \frac{x-2(1-\hbar\omega_D/2-\mu)}{2T_c} - 1] = n. \quad (76)$$

which are the gap-like equation associated with 2h-CPs and its corresponding number equation. Note that (74) and (76) are quadratic in n . Further, all integrals there are exact, i.e.,

$$\int_0^\infty dx [1 - \tanh \frac{x-\mu}{2T_c}] = \mu + 2T_c \ln[2 \cosh(\mu/2T_c)], \quad (77)$$

and

$$\int_0^\infty dx x [\coth \frac{x+\delta}{2T_c} - 1] = 2T_c^2 \text{PolyLog}[2, e^{-\delta/T_c}] \quad (78)$$

where the polylogarithm function as defined in Ref. [114] is

$$\text{PolyLog}[\sigma, z] \equiv \sum_{l=1}^{\infty} z^l / l^\sigma. \quad (79)$$

The integrals in (73) and (75), however, were performed numerically. In 2D we use the two extreme values of $\lambda = 1/4$ (lower set of curves in Figs. 10 and 11) and $\lambda = 1/2$ (upper set of curves), and $\hbar\omega_D/E_F = 0.05$ (a typical value for cuprates). If $\lambda > 1/2$ the ionic lattice in 3D becomes unstable, [115] and Ref. [3] p. 204. Equations (73) to (76) lead to the T_c/T_F vs. n/n_f phase diagram which is graphed in Figs. 10 and 11 for both 2e-CP (dashed curve) and 2h-CP (full curve) pure, stable BEC-like phases. The value $n/n_f = 1$ corresponds to perfect 2e/2h-CP symmetry for the lower- T_c mixed phase (not shown), while $n/n_f > 1$ (and < 1) signifies more (less) 2e-CPs than 2h-CPs in the mixed phase. The T_c value where both phase-boundary curves $n_0(T_c) = m_0(T_c) = 0$ intersect is marked by the large dots in the figure. The values are consistent with those gotten from (68) which gives $\simeq 0.001$ for $\lambda = 1/4$ and $\simeq 0.008$ for $\lambda = 1/2$, for $\hbar\omega_D/E_F = 0.05$. [The values from (68) hardly differ from those from the exact BCS (implicit) T_c -formula (Ref. [44], p. 447) $1 = \lambda \int_0^{\hbar\omega_D/2k_B T_c} dx x^{-1} \tanh x$.] Cuprate data empirically [116] show T_c 's and T_F 's falling within the range $T_c/T_F \simeq 0.03 - 0.09$. Thus, moderate departures from perfect 2e/2h-CP symmetry enable the CBFM to predict quasi-2D cuprate empirical T_c values, *without abandoning electron-phonon dynamics*—contrary to popular belief.

As the variable n_f used to scale the electron number density n in Fig. 10 does not appear at all in the number equation (57), we also scaled n with the variable $n_f(T)$ evaluated at $T = T_c$ which does appear in (57) and which has a clear-cut physical meaning. Since $\Delta(T_c) = 0$, from (51) and (52), (60) reduces in 2D to the analytic expression

$$n_f(T_c)/n \equiv (T_c/T_F) \ln[1 + \exp\{\mu(T_c)/k_B T_c\}]. \quad (80)$$

For the pure 2e-CP condensate case, i.e., $m_B(T) \equiv 0$, we must have $n/n_f(T_c) \geq 1$ from (57), where in general $n_f(T_c) \neq n_f$. Here $n_f(T_c)$ is the actual number density of unpaired electrons at $T = T_c$ since the number equation (57) for $m_B(T) = 0$ then becomes just

$$n_f(T_c) + 2n_{B+}(T_c) = n. \quad (81)$$

Here $n_{B+}(T_c)$ is the total number density of all *excited* (or $K > 0$) 2e-CPs, also commonly known as “preformed CPs.” It is defined in (58) at any T as

$$n_{B+}(T) \equiv \int_{0^+}^{\infty} d\eta M(\eta) \frac{1}{e^{\beta \mathcal{E}_+(\eta)} - 1}. \quad (82)$$

Preformed CPs seem to be the main conjecture to explain a “pseudogap” [117] observed above T_c in many underdoped cuprates.

The complete number equation (57) then explicitly reads

$$2n_0(T) + 2n_{B+}(T) - 2m_0(T) - 2m_{B+}(T) + n_f(T) = n \quad (83)$$

with $m_{B+}(T)$ being the number of preformed 2h-CPs, defined as the integral term in (59). Using $n_f(T_c)$ instead of n_f to scale electron densities n in (73) to (76) which are solved numerically, scaling all energies with E_F instead of E_f as before, and on eliminating n_f , gives the results in Fig. 11. Curiously, these curves are hard to distinguish from those of Fig. 10. In Fig. 11 the BCS values (black dots in figure) are *not* associated as before with the intersection of the 2e- and 2h-CP condensation curves, but are merely values calculated with the BCS T_c formula (68). On the other hand, the T_c values from the CBFM equations scaled with the previous (and somewhat opaque) variable n_f , match only *approximately* in the limit $n/n_f \rightarrow \infty$ with the analytical BEC limits of (71). It has been verified numerically that coincidence in the BEC limit is perfect only in the trivial limit of zero coupling, as expected.

In view of Figs. 10 and 11 we note that *room temperature superconductivity* (RTSC in figures) is possible, e.g., in Fig. 11, for $n/n_f(T_c)$ sufficiently less than unity. From (83) with $n_B(T) \equiv n_0(T) + n_{B+}(T) \equiv 0$ this translates into this recipe: $2m_{B+}(T_c)$ must be sufficiently larger than $n_f(T_c)$. Specifically, from Fig. 11 RTSC occurs for $\lambda = \frac{1}{2}$ if $n/n_f(T_c) \lesssim 0.1$, which from (57) means $2m_{B+}(T_c) \gtrsim 0.9n_f(T_c)$ or that the total number of individual holes bound into *preformed* 2h-CPs at T_c must somehow be larger than about 90% of the number of unpaired electrons at T_c .

8.2 Three dimensions (3D)

In 3D the electronic DOS if $\epsilon \equiv \hbar^2 k^2/2m$ is the familiar expression

$$N(\epsilon) \equiv (1/2\pi^2)k^2(dk/d\epsilon) = (m^{3/2}/2^{1/2}\pi^2\hbar^3)\sqrt{\epsilon} \quad (84)$$

which coincides with (48). Analogously as before, $E_f = (\hbar^2/2m)(3\pi^2 n_f)^{2/3} \equiv k_B T_f$ again differs from $E_F = (\hbar^2/2m)(3\pi^2 n)^{2/3} \equiv k_B T_F$ unless perfect 2e/2h-CP symmetry holds in which case they coincide. On the other hand, the leading term in the BS CP boson dispersion energy is now the linear expression (70) so that

$$M(\eta) \equiv (1/2\pi^2)K^2(dK/d\eta) \simeq (32/\pi^2\lambda^3\hbar^3 v_F^3)\eta^2. \quad (85)$$

The above working equations for the *pure 2e-CP phase* in 3D (all quantities dimensionless as before in terms of E_f and n_f) are

$$1 + \hbar\omega_D/2 - \mu = \lambda(\hbar\omega_D/2) \frac{1}{n^{1/3}} \int_1^{1+\hbar\omega_D} dx \sqrt{x} \frac{1}{|x - \mu|} \tanh \frac{|x - \mu|}{2T_c}, \quad (86)$$

$$\frac{3}{4} \int_0^\infty dx \sqrt{x} [1 - \tanh \frac{x - \mu}{2T_c}] + \frac{12}{\lambda^3} \frac{1}{n} \int_0^\infty dx x^2 [\coth \frac{x + 2(1 + \hbar\omega_D/2 - \mu)}{2T_c} - 1] = n, \quad (87)$$

while for the *pure 2h-CP phase* they are

$$\mu - 1 + \hbar\omega_D/2 = \lambda(\hbar\omega_D/2) \frac{1}{n^{1/3}} \int_{1-\hbar\omega_D}^1 dx \sqrt{x} \frac{1}{|x - \mu|} \tanh \frac{|x - \mu|}{2T_c}, \quad (88)$$

$$\frac{3}{4} \int_0^\infty dx \sqrt{x} [1 - \tanh \frac{x - \mu}{2T_c}] - \frac{12}{\lambda^3} \frac{1}{n} \int_0^\infty dx x^2 [\coth \frac{x - 2(1 - \hbar\omega_D/2 - \mu)}{2T_c} - 1] = n. \quad (89)$$

Results in 3D are reported only for $\lambda = 1/2$, and $\hbar\omega_D/E_F = 0.005$ was used here. In Figs. 12 and 13 the full curves are again the 2h-CP BEC phase boundaries while the dashed curves are the 2e-CP BEC ones. The large dot again marks the BCS T_c/T_F values obtainable from (68) of 0.0001 for $\lambda = 1/4$ and of 0.0008 for $\lambda = 1/2$. Besides Ref. [116], empirical data for both exotic and conventional, elemental superconductors in 3D are also graphed in Ref. [118]. We see that whereas BCS theory roughly reproduces T_c/T_F values well for the latter SCs, it takes moderate departures from perfect 2e/2h-CP symmetry to access 3D exotic SC T_c/T_F values, which empirically fall within the range $0.01 - 0.1$. This is much larger than the range $\lesssim 0.001$ for conventional (elemental) superconductors, also shaded in the figure.

9 Hole superconductivity

Finally, we address the unique but mysterious role played by *hole* charge carriers [46] in the normal state of superconductors in general. For example: a) of the cuprates, those that are hole-doped have transition temperatures T_c about *six* times higher than electron-doped ones; and even in conventional superconductors [47] b) over 80% of all superconducting elements have positive Hall coefficients (meaning hole charge carriers); while c) over 90% of non-superconducting metallic, non-magnetic elements have electron charge carriers. This greater “efficiency” of individual, unpaired hole carriers in producing higher T_c s is clearly reflected in Figs. 10 and 11 for both 2D and 3D superconductors, at least insofar as pure 2h-CP BE condensates exhibiting higher T_c s than those associated with pure 2e-CP BE condensates.

10 Conclusions

This review began with a survey of “ordinary” and “generalized” Cooper pairing, and stressed that if hole pairing is treated on an equal footing with electron pairing the original “ordinary” Cooper problem (based on the pure-kinetic-energy unperturbed Hamiltonian) is meaningless. However, “generalized” Cooper pairs defined in terms of the Bethe-Salpeter equation including both electron and hole pairs, in the ladder approximation, but based on the BCS ground state as unperturbed Hamiltonian instead of on the ideal Fermi gas sea, gives rise to physically meaningful positive-energy resonances with a finite lifetime for CMM $K > 0$ and infinite lifetime for $K = 0$.

It was then sketched how five statistical continuum theories of superconductivity—including both the BCS and BEC theories—are contained as special limiting cases within a single theory, the “complete boson-fermion model” (CBFM). This model includes, for the first time, both two-electron and two-hole pairs in freely variable proportions, along with unpaired electrons. Thus, the BCS and BEC theories are fully *unified* within the CBFM. The BCS condensate [specified by a single equation, namely the T -dependent gap equation for $\Delta(T)$] follows directly from the CBFM *as a BE condensate* through the condition for phase equilibria when both total 2e- and 2h-pair number, as well as their condensate, densities are *equal* at a given temperature and coupling—provided the coupling is weak enough so that the electron chemical potential μ roughly equals the Fermi energy E_F . Ordinary BEC theory, on the other hand, is recovered from the CBFM when hole pairs are completely neglected, the BF coupling f is made to vanish, and the limit of zero unpaired electrons is taken.

Lastly, the practical outcome of the BCS-BEC unification via the CBFM is fourfold: a) *enhancements* in T_c , by more than an order-of-magnitude in 2D, and more than two orders-of-magnitude in 3D, are obtained with pure electron-pair, and even more so with pure hole-pair, BE condensates for the same electron-phonon dynamics mimicked by the BCS model interaction; b) these enhancements in T_c fall within empirical ranges for 2D and 3D “exotic” SCs, whereas BCS T_c values remain low and within the empirical ranges for conventional, elemental SCs using standard interaction-parameter values; c) hole-doped SCs are predicted in both 2D and 3D to have higher T_c ’s than electron-doped ones, in agreement with observation; and finally that d) room temperature superconductivity is possible, with the *same* interaction parameters used in BCS theory for conventional SCs, but only via hole-pair BE condensates.

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Figure Captions

1. Dimensionless CP excitation energy $\varepsilon_K/(-\mathcal{E}_0) \equiv \varepsilon_K/\Delta_0$ vs K/k_F , numerically determined from (8) for different couplings B_2/E_F , full curves, for the delta interaction in 2D. The dot-dashed line is the linear approximation (virtually coincident with the exact curve for $B_2/E_F \lesssim 0.1$) while the dashed curve is the pure quadratic term of (9). Dots denote values of CMM wavenumber where the CP breaks up, i.e., where $\mathcal{E}_K \equiv 0$.
2. Cross-section of overlap volume in k -space (shading) where the tip of the relative CP wavevector $\mathbf{k} \equiv \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2)$ must point for the attractive BCS model interaction (15) to be nonzero for a CP of CMM $\hbar\mathbf{K} \equiv \hbar(\mathbf{k}_1 + \mathbf{k}_2)$.
3. Wavefunction Feynman diagrams for 2p (ψ_+), 2h (ψ_-) and ph (ψ_0) bound states arising from the BS equations. Shaded rectangles designate diagrams that do not contribute in the IFG-based case.
4. a) Exact “moving” 2p-CP (real) energy \mathcal{E}_K (in units of E_F) in 2D from (37) (full curves), compared with its linear leading term (thin short-dashed lines) and its linear plus quadratic expansion (long-dashed curves) both from (38), vs. CMM wavenumber K (in units of k_F), for interaction (20) parameters $\lambda = \frac{1}{4}$ (lower set of curves) and $\frac{1}{2}$ (upper set of curves), and $\hbar\omega_D/E_F = 0.05$. For reference, leading linear term (34) of trivial ABH sound mode is also plotted (lower thick dashed line). b) 2p-CP lifetime as defined in (39). c) Analogy of ordinary and BCS-based-BS 2p-CPs with various confined states in a 3D potential problem.
5. BE condensate-fraction curves $1 - (T/T_c)^{d/s}$ for bosons in $d = 3, 2$, or 1 with dispersion relation (1) with $s = 2$ or 1 , for a *pure* phase of either 2e- or 2h-CPs as discussed in text, compared to empirical data for 3D SCs (*Nb/Cu* and *Sn*), two quasi-2D SCs (*Y123* and *Bi2212*) and a quasi-1D SC (4 *A*-wide nanotubes). Data for the latter are for $\Delta(T)/\Delta(0)$ but are plotted as $[\Delta(T)/\Delta(0)]^2$ so as to reflect 2h-CP condensate fraction $m_0(T)/m_0(0)$ according to (67). The curve marked 1/1 strictly corresponds to $T_c \equiv 0$; however, it serves as a lower bound to all curves with $d/s = (1 + \epsilon)/1$ for small but nonzero ϵ for which $T_c > 0$. The ordinate axis is labeled with the superelectron number density $n_s(T)$ in units of the normal electron density n . Also shown for reference are the two-fluid model [80] curve $1 - (T/T_c)^4$ and the BCS gap $\Delta(T)/\Delta(0)$ order parameter curve [81].
6. BF vertex diagrams depicting disintegration of 2e-CPs into two unpaired electrons; 2e-CP formation process from the latter; formation of a 2h-CP from two unpaired holes; and the disintegration of a 2h-CP into the latter. All four processes are contained in H_{int} Eq. (42).
7. Flow chart of how the CBFM reduces in special cases to the statistical, continuum models of superconductivity discussed in text, thereby displaying how both BCS and BEC theories are unified. The three “legs” refer to: perfect 2e-/2h-CP symmetry (middle leg, giving rise to the BCS-Bose crossover and BCS theories); to

no 2h-CPs present (right leg) leading to ordinary BEC as a special case; and to no 2e-CPs present (left leg) predicting room temperature superconductivity (RTSC), see Figs. 10 and 13 below, via a 2h-CP BE condensate.

8. Critical 2D BEC-like temperature (T_c in units of T_F) approached from *above* within the ideal BF model (IBFM) with the BCS model interaction for $\lambda = 1/2$ for varying $\hbar\omega_D/E_F \equiv \Theta_D/T_F$. Results are for: the pure unbreakable-boson gas with *some* and with *all* fermions paired; for the breakable-boson gas; and for the boson-fermion mixture in thermal/chemical equilibrium (thick full curve labeled “binary gas”), all as described in Ref. [18] for original (simple) CPs where $C_1 = (2/\pi)\hbar v_F$ in (1). Dashed curve is the BCS theory T_c , given by (68). Cuprate experimental data are taken from Ref. [116]. The IBFM can be considered a model for the normal state, and its temperature instability a prediction of T_c .

9. Phase diagram [14] with 3D superconducting critical temperature phase boundaries T_{cs+} , T_{cs-} , T_{css+} , and T_{css-} as functions of $\Delta n \equiv n/n_f - 1$ as defined in text, in the vicinity of the approximate BCS T_c value, assuming the quadratic boson dispersion $\eta = \hbar^2 K^2/2(2m)$, for $\lambda = 1/5$ and $\hbar\omega_D/E_F = 0.001$.

10. Phase diagram in 2D temperature T (in units of T_F) and electron density n (in units of n_f as defined in text) showing the phase boundaries of T_c 's for pure 2e-CP BEC phases (dashed curves) determined by $\Delta(T_c) = f\sqrt{n_0(T_c)} \equiv 0$ and pure 2h-CP BEC phases (full curves) given by $\Delta(T_c) = f\sqrt{m_0(T_c)} \equiv 0$ for $\lambda = 1/4$ and $1/2$ with $\hbar\omega_D/E_F = 0.05$. Intersections corresponding to $n_0(T) = m_0(T)$ approximately reproduce the BCS T_c as given by (67) and are marked by black dots. Black squares mark the BEC limit where all electrons are imagined paired into 2e-CP bosons, as calculated in (71), to which values the dashed curves reach only approximately in the limit $n/n_f \rightarrow \infty$.

11. Same as Fig. 10 except that n/n_f is eliminated numerically and abscissa refers to $n/n_f(T_c)$, where $n_f(T_c)$ is the actual number of unpaired electrons at T_c as given by (80). Here, the intersection corresponding to $n_0(T) = m_0(T)$ occurs at $T = 0$ and *not* at the approximate BCS T_c values as in Fig. 10. Open circles mark values of $n/n_f(T_c)$ corresponding to $n/n_f = \infty$.

12. Same as Fig. 10 but in 3D, and only for $\lambda = \frac{1}{2}$.

13. Same as Fig. 11 but in 3D, and only for $\lambda = \frac{1}{2}$.

Appendix A. Why CPs are bosons while BCS pairs are not.

This assertion is simply visualized *qualitatively* with the aid of Fig. 2 for the BCS model interaction (15). The vector \mathbf{k} ends in all points of a simple-cubic lattice in k -space with lattice spacing $2\pi/L$ with L the system size. In 3D these points are within the overlap volume (shaded in the figure) where the interaction is nonzero. But, in the thermodynamic limit there are infinitely many acceptable \mathbf{k} values allowed for the energy \mathcal{E}_K of the CP state, whether for ordinary CPs (7), (16) or for Bethe-Salpeter CPs (37). This implies BE statistics for either ordinary or BS CPs as each CP energy \mathcal{E}_K level has no occupation limit. Not so for BCS pairs as we now show.

More *quantitatively*, consider fermion creation $a_{\mathbf{k}_1 s}^\dagger$ and annihilation $a_{\mathbf{k}_1 s}$ operators that satisfy the anti-commutation relations.

$$\begin{aligned} \{a_{\mathbf{k}_1 s}^\dagger, a_{\mathbf{k}'_1 s'}^\dagger\} &= \{a_{\mathbf{k}_1 s}, a_{\mathbf{k}'_1 s'}\} = 0 \\ \{a_{\mathbf{k}_1 s}, a_{\mathbf{k}'_1 s'}^\dagger\} &= \delta_{\mathbf{k}_1 \mathbf{k}'_1} \delta_{ss'}. \end{aligned} \quad (\text{A.1})$$

The BCS pair annihilation and creation operators, respectively, are then

$$b_{\mathbf{K}} \equiv a_{\mathbf{k}_2 \downarrow} a_{\mathbf{k}_1 \uparrow} \quad \text{and} \quad b_{\mathbf{K}}^\dagger \equiv a_{\mathbf{k}_1 \uparrow}^\dagger a_{\mathbf{k}_2 \downarrow}^\dagger. \quad (\text{A.2})$$

Here

$$\mathbf{k} \equiv \frac{1}{2}(\mathbf{k}_1 - \mathbf{k}_2) \quad \text{and} \quad \mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2 \quad (\text{A.3})$$

are the relative and total (or center-of-mass) momenta wavevectors, respectively, associated with two fermions with wavevectors

$$\mathbf{k}_1 = \mathbf{K}/2 + \mathbf{k} \quad \text{and} \quad \mathbf{k}_2 = \mathbf{K}/2 - \mathbf{k}. \quad (\text{A.4})$$

We show below that $b_{\mathbf{k}\mathbf{K}}$ and $b_{\mathbf{k}'\mathbf{K}}^\dagger$ satisfy: a) the sometimes called “pseudo-boson” commutation relations

$$[b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}^\dagger] = (1 - n_{\mathbf{K}/2-\mathbf{k}\downarrow} - n_{\mathbf{K}/2+\mathbf{k}\uparrow})\delta_{\mathbf{k}\mathbf{k}'}, \quad (\text{A.5})$$

$$[b_{\mathbf{k}\mathbf{K}}^\dagger, b_{\mathbf{k}'\mathbf{K}}^\dagger] = [b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}] = 0, \quad (\text{A.6})$$

where

$$n_{\mathbf{K}/2\pm\mathbf{k}\downarrow} \equiv a_{\mathbf{K}/2\pm\mathbf{k}\downarrow}^\dagger a_{\mathbf{K}/2\pm\mathbf{k}\downarrow} \quad (\text{A.7})$$

are fermion number operators; as well as b) a “pseudo-fermion” anti-commutation relation

$$\{b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}\} = 2b_{\mathbf{k}\mathbf{K}}b_{\mathbf{k}'\mathbf{K}}(1 - \delta_{\mathbf{k}\mathbf{k}'}). \quad (\text{A.8})$$

Our only restriction is that $\mathbf{K} \equiv \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}'_1 + \mathbf{k}'_2$. Clearly, BCS pairs are *not* bosons as they do *not* satisfy (Ref. [34] p. 38) the ordinary boson commutation relations

$$[b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'} \quad \text{and} \quad [b_{\mathbf{k}\mathbf{K}}^\dagger, b_{\mathbf{k}'\mathbf{K}}^\dagger] = [b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}] = 0. \quad (\text{A.9})$$

If $\mathbf{K} = 0$ (so that $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$), and calling $b_{\mathbf{k}\mathbf{K}=0} \equiv b_{\mathbf{k}}$, etc., (A.5) and (A.6) become Eqs. (2.11) and (2.12) of Ref. [7], and (A.8) becomes Eq. (2.13) thereof.

To prove (A.5) to (A.8) write

$$[b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}^\dagger] \equiv b_{\mathbf{k}\mathbf{K}}b_{\mathbf{k}'\mathbf{K}}^\dagger - b_{\mathbf{k}'\mathbf{K}}^\dagger b_{\mathbf{k}\mathbf{K}} \equiv a_{\mathbf{k}_2\downarrow}a_{\mathbf{k}_1\uparrow}a_{\mathbf{k}'_1\uparrow}^\dagger a_{\mathbf{k}'_2\downarrow}^\dagger - a_{\mathbf{k}'_1\uparrow}^\dagger a_{\mathbf{k}'_2\downarrow}^\dagger a_{\mathbf{k}_2\downarrow}a_{\mathbf{k}_1\uparrow} \quad (\text{A.10})$$

using (A.2). This can alternately be rewritten, using (A.4), as

$$[b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}^\dagger] = a_{\mathbf{K}/2-\mathbf{k}\downarrow}a_{\mathbf{K}/2+\mathbf{k}\uparrow}a_{\mathbf{K}/2+\mathbf{k}'\uparrow}^\dagger a_{\mathbf{K}/2-\mathbf{k}'\downarrow}^\dagger - a_{\mathbf{K}/2+\mathbf{k}'\uparrow}^\dagger a_{\mathbf{K}/2-\mathbf{k}'\downarrow}^\dagger a_{\mathbf{K}/2-\mathbf{k}\downarrow}a_{\mathbf{K}/2+\mathbf{k}\uparrow}. \quad (\text{A.11})$$

The pair of Fermi operators $a_{\mathbf{K}/2+\mathbf{k}\uparrow}a_{\mathbf{K}/2+\mathbf{k}'\uparrow}^\dagger$ in the first term gives, using (A.1),

$$\begin{aligned} & a_{\mathbf{K}/2-\mathbf{k}\downarrow}(\delta_{\mathbf{k}\mathbf{k}'} - a_{\mathbf{K}/2+\mathbf{k}'\uparrow}^\dagger a_{\mathbf{K}/2+\mathbf{k}\uparrow})a_{\mathbf{K}/2-\mathbf{k}'\downarrow}^\dagger \\ &= a_{\mathbf{K}/2-\mathbf{k}\downarrow}a_{\mathbf{K}/2-\mathbf{k}'\downarrow}^\dagger \delta_{\mathbf{k}\mathbf{k}'} - a_{\mathbf{K}/2-\mathbf{k}\downarrow}a_{\mathbf{K}/2+\mathbf{k}'\uparrow}^\dagger a_{\mathbf{K}/2+\mathbf{k}\uparrow}a_{\mathbf{K}/2-\mathbf{k}'\downarrow}^\dagger. \end{aligned} \quad (\text{A.12})$$

Using (A.1) again the first term becomes $\delta_{\mathbf{k}\mathbf{k}'}(1 - a_{\mathbf{K}/2-\mathbf{k}'\downarrow}^\dagger a_{\mathbf{K}/2-\mathbf{k}\downarrow})$. The last term of (A.12), after anti-commuting all creation operators to the left and recalling that $\delta_{\uparrow\downarrow} \equiv 0$, etc., gives $-\delta_{\mathbf{k}\mathbf{k}'}a_{\mathbf{K}/2+\mathbf{k}'\uparrow}^\dagger a_{\mathbf{K}/2+\mathbf{k}\uparrow} + a_{\mathbf{K}/2+\mathbf{k}'\uparrow}^\dagger a_{\mathbf{K}/2-\mathbf{k}'\downarrow}^\dagger a_{\mathbf{K}/2-\mathbf{k}\downarrow}a_{\mathbf{K}/2+\mathbf{k}\uparrow}$. Inserting this in (A.11) leaves precisely (A.5) if we recall the number operators (A.7). To prove (A.6) write

$$[b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}] \equiv b_{\mathbf{k}\mathbf{K}}b_{\mathbf{k}'\mathbf{K}} - b_{\mathbf{k}'\mathbf{K}}b_{\mathbf{k}\mathbf{K}} \equiv a_{\mathbf{k}_2\downarrow}a_{\mathbf{k}_1\uparrow}a_{\mathbf{k}'_2\downarrow}a_{\mathbf{k}'_1\uparrow} - a_{\mathbf{k}'_2\downarrow}a_{\mathbf{k}'_1\uparrow}a_{\mathbf{k}_2\downarrow}a_{\mathbf{k}_1\uparrow} \quad (\text{A.13})$$

using (A.2); or equivalently

$$[b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}] \equiv a_{\mathbf{K}/2-\mathbf{k}\downarrow}a_{\mathbf{K}/2+\mathbf{k}\uparrow}a_{\mathbf{K}/2-\mathbf{k}'\downarrow}a_{\mathbf{K}/2+\mathbf{k}'\uparrow} - a_{\mathbf{K}/2-\mathbf{k}'\downarrow}a_{\mathbf{K}/2+\mathbf{k}'\uparrow}a_{\mathbf{K}/2-\mathbf{k}\downarrow}a_{\mathbf{K}/2+\mathbf{k}\uparrow}. \quad (\text{A.14})$$

The first term is easily brought into a form cancelling the last term by simply anticommuting operators with primed subindices to the left, thus proving (A.6). Finally to prove (A.8) write

$$\{b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}'\mathbf{K}}\} \equiv b_{\mathbf{k}\mathbf{K}}b_{\mathbf{k}'\mathbf{K}} + b_{\mathbf{k}'\mathbf{K}}b_{\mathbf{k}\mathbf{K}} \equiv a_{\mathbf{k}_2\downarrow}a_{\mathbf{k}_1\uparrow}a_{\mathbf{k}'_2\downarrow}a_{\mathbf{k}'_1\uparrow} + a_{\mathbf{k}'_2\downarrow}a_{\mathbf{k}'_1\uparrow}a_{\mathbf{k}_2\downarrow}a_{\mathbf{k}_1\uparrow} \quad (\text{A.15})$$

$$\equiv a_{\mathbf{K}/2-\mathbf{k}\downarrow} a_{\mathbf{K}/2+\mathbf{k}\uparrow} a_{\mathbf{K}/2-\mathbf{k}'\downarrow} a_{\mathbf{K}/2+\mathbf{k}'\uparrow} + a_{\mathbf{K}/2-\mathbf{k}'\downarrow} a_{\mathbf{K}/2+\mathbf{k}'\uparrow} a_{\mathbf{K}/2-\mathbf{k}\downarrow} a_{\mathbf{K}/2+\mathbf{k}\uparrow}, \quad (\text{A.16})$$

$$= 2a_{\mathbf{K}/2-\mathbf{k}\downarrow} a_{\mathbf{K}/2+\mathbf{k}\uparrow} a_{\mathbf{K}/2-\mathbf{k}'\downarrow} a_{\mathbf{K}/2+\mathbf{k}'\uparrow} \equiv 2b_{\mathbf{k}\mathbf{K}} b_{\mathbf{k}'\mathbf{K}} \quad \text{if } \mathbf{k} \neq \mathbf{k}'. \quad (\text{A.17})$$

However, if $\mathbf{k} = \mathbf{k}'$ then

$$\{b_{\mathbf{k}\mathbf{K}}, b_{\mathbf{k}\mathbf{K}}\} = 2a_{\mathbf{K}/2-\mathbf{k}\downarrow} a_{\mathbf{K}/2+\mathbf{k}\uparrow} a_{\mathbf{K}/2-\mathbf{k}\downarrow} a_{\mathbf{K}/2+\mathbf{k}\uparrow} = 0 \quad (\text{A.18})$$

since $(a_{\mathbf{K}/2\pm\mathbf{k}\downarrow})^2 \equiv 0$. Hence (A.8) is true.

The main point here is simply this: *any* number of CPs with a definite \mathbf{K} but *different* values of \mathbf{k} can occupy a state of CP energy \mathcal{E}_K and thus not only obey BE statistics but (as this in turn demands) also obey the boson commutation relations (A.5) for $\mathbf{k} \neq \mathbf{k}'$. Hence, CPs can suffer a BEC as a given CP state involves no two BCS pairs with the same \mathbf{k} .

Appendix B. CBFM and BCS condensation energies compared.

Using (47) for $T = 0$ when $n_0(T) = m_0(T)$ and $n_B(T) = m_B(T)$ we obtain for the superconducting state, calling $\Delta(T = 0) \equiv \Delta$,

$$\frac{\Omega_s(T=0)}{L^d} = 2\hbar\omega_D n_0(0) + \int_{-\mu}^{\infty} d\xi N(\xi) [\xi - \sqrt{\xi^2 + \Delta^2}]$$

where $\xi \equiv \varepsilon - \mu$. Recalling the expression (52) for Δ , using (43) and (44), and since $\mu = E_F$, gives

$$\frac{\Omega_s(T=0)}{L^d} = 2\hbar\omega_D n_0(0) + 2 \int_{-\mu}^{-\hbar\omega_D} d\xi N(\xi) \xi + N(E_F) \int_{-\hbar\omega_D}^{\hbar\omega_D} d\xi [\xi - \sqrt{\xi^2 + \Delta^2}]. \quad (\text{B.1})$$

The first and second members of the last term have respectively odd and even integrands, so that this term becomes

$$-2N(E_F) \int_0^{\hbar\omega_D} d\xi [\sqrt{\xi^2 + \Delta^2}].$$

On the other hand, for the normal state $n_0 = 0$ and $m_0 = 0$, so that from (52) $\Delta(T) = 0$. Hence, from (64) the CBFM condensation energy $E_s - E_n$ per unit volume is just

$$\frac{E_s - E_n}{L^d} = 2\hbar\omega_D n_0(0) + 2N(E_F) \int_0^{\hbar\omega_D} d\xi (\xi - \sqrt{\xi^2 + \Delta^2}) \quad (\text{CBFM}). \quad (\text{B.2})$$

From Eq. (2), Sec. 4.3.3.1, Ref. [119] the integral evaluates to

$$\begin{aligned} & -\frac{1}{2}\hbar\omega_D \sqrt{(\hbar\omega_D)^2 + \Delta^2} + \frac{(\hbar\omega_D)^2}{2} + \frac{1}{2}\Delta^2 \ln \frac{\Delta}{\hbar\omega_D + \sqrt{(\hbar\omega_D)^2 + \Delta^2}} \\ & = \frac{1}{2}\Delta^2 \ln(\Delta/2\hbar\omega_D) - \frac{1}{4}\Delta^2 - \frac{1}{16}[\Delta^4/(\hbar\omega_D)^2] + O[\Delta^6/(\hbar\omega_D)^4]. \end{aligned}$$

Thus, since $n_0 = \Delta^2/f^2$, (B.2) for weak coupling $\Delta = f\sqrt{n_0} \rightarrow 0$ becomes

$$\frac{E_s - E_n}{L^d} \xrightarrow{\lambda \rightarrow 0} -\frac{1}{2}N(E_F)\Delta^2 \left[1 + \frac{1}{4}(\Delta/\hbar\omega_D)^2 + O(\Delta/\hbar\omega_D)^4 \right] \quad (\text{CBFM}). \quad (\text{B.3})$$

By contrast, the original expressions (2.41) and (2.43) in Ref. [7] give

$$\frac{E_s - E_n}{L^d} = 2N(E_F) \int_0^{\hbar\omega_D} d\xi \left(\xi - \frac{\xi^2}{\sqrt{\xi^2 + \Delta^2}} \right) - \frac{\Delta^2}{V} \quad (\text{BCS}) \quad (\text{B.4})$$

where V is defined in (15). Multiplying (32) by $\Delta^2/2$ is equivalent to

$$2N(E_F) \int_0^{\hbar\omega_D} d\xi \frac{\Delta^2}{2\sqrt{\xi^2 + \Delta^2}} = \frac{\Delta^2}{V}$$

which when combined with (B.4) gives

$$\frac{E_s - E_n}{L^d} = 2N(E_F) \int_0^{\hbar\omega_D} d\xi \left[\xi - \frac{1}{2} \frac{2\xi^2 + \Delta^2}{\sqrt{\xi^2 + \Delta^2}} \right].$$

Using Eqs. (3) and (10), Sec. 4.3.3.1, of Ref. [119] finally gives

$$\frac{E_s - E_n}{L^d} = N(E_F)(\hbar\omega_D)^2 \left[1 - \sqrt{1 + (\Delta/\hbar\omega_D)^2} \right] \quad (\text{B.5})$$

which on expansion leads to

$$\frac{E_s - E_n}{L^d} \xrightarrow{\lambda \rightarrow 0} -\frac{1}{2}N(E_F)\Delta^2 \left[1 - \frac{1}{4}(\Delta/\hbar\omega_D)^2 + O(\Delta/\hbar\omega_D)^4 \right] \quad (\text{BCS}). \quad (\text{B.6})$$

Thus, the CBFM condensation energy (B.3) is *larger* than that of BCS, but only the latter is a rigorous upper bound to the exact condensation energy since it follows [7] from a variational trial wavefunction.

Appendix C. BEC as limit of all electrons paired.

Here we discuss generalizations of the well-known result that the BEC transition temperature T_c of a d -dimensional N -fermion gas of Fermi temperature T_F in which *all* fermions are imagined paired into bosons, is just $0.218T_F$. (See dashed line in “Uemura plot” of Ref. [118], Fig. 2). These results will provide a convenient pure BEC limit which in effect turn out to be an *upper limit* for the pure 2e-CP phase separation boundary T_c values deduced in Sec. 8.

The general BEC T_c -formula for identical noninteracting bosons in d -dimensions of energy $\eta = C_s K^s$, $s > 0$, where as before K is the boson CMM, is [16]

$$T_c = \frac{C_s}{k_B} \left[\frac{s\Gamma(d/2)(2\pi)^d}{2\pi^{d/2}\Gamma(d/s)g_{d/s}(1)} n_B \right]^{s/d} \quad (\text{C.1})$$

where n_B is the boson number-density and $g_\sigma(z)$ the Bose integral [120], $z \equiv e^{\mu_B/k_B T}$ is the “*fugacity*” and μ_B the boson chemical potential. For $z = 1$, $g_\sigma(1) \equiv \zeta(\sigma)$, the Riemann Zeta-function, if $\sigma > 1$, while for $0 < \sigma \leq 1$ the infinite series $g_\sigma(1)$ diverges. Eq. (C.1) is formally valid for all $d > 0$ and $s > 0$. Hence, for $0 < d \leq s$, $T_c = 0$ since $g_{d/s}(1) = \infty$ for $d/s \leq 1$ but T_c is otherwise finite. We stress that as a consequence of the former *all* 2D T_c predictions in Fig. 7 (except the BCS one that survives for all $d > 0$, including $d = 1$) would collapse to *zero* had $s = 2$ been used in 2D instead of the correct $s = 1$ arising from the Fermi sea. For $s = 2$, $C_2 = \hbar^2/2m_B$ (C.1) leads to the familiar 3D result

$$T_c \simeq 3.31\hbar^2 n_B^{2/3} / m_B k_B \quad (\text{C.2})$$

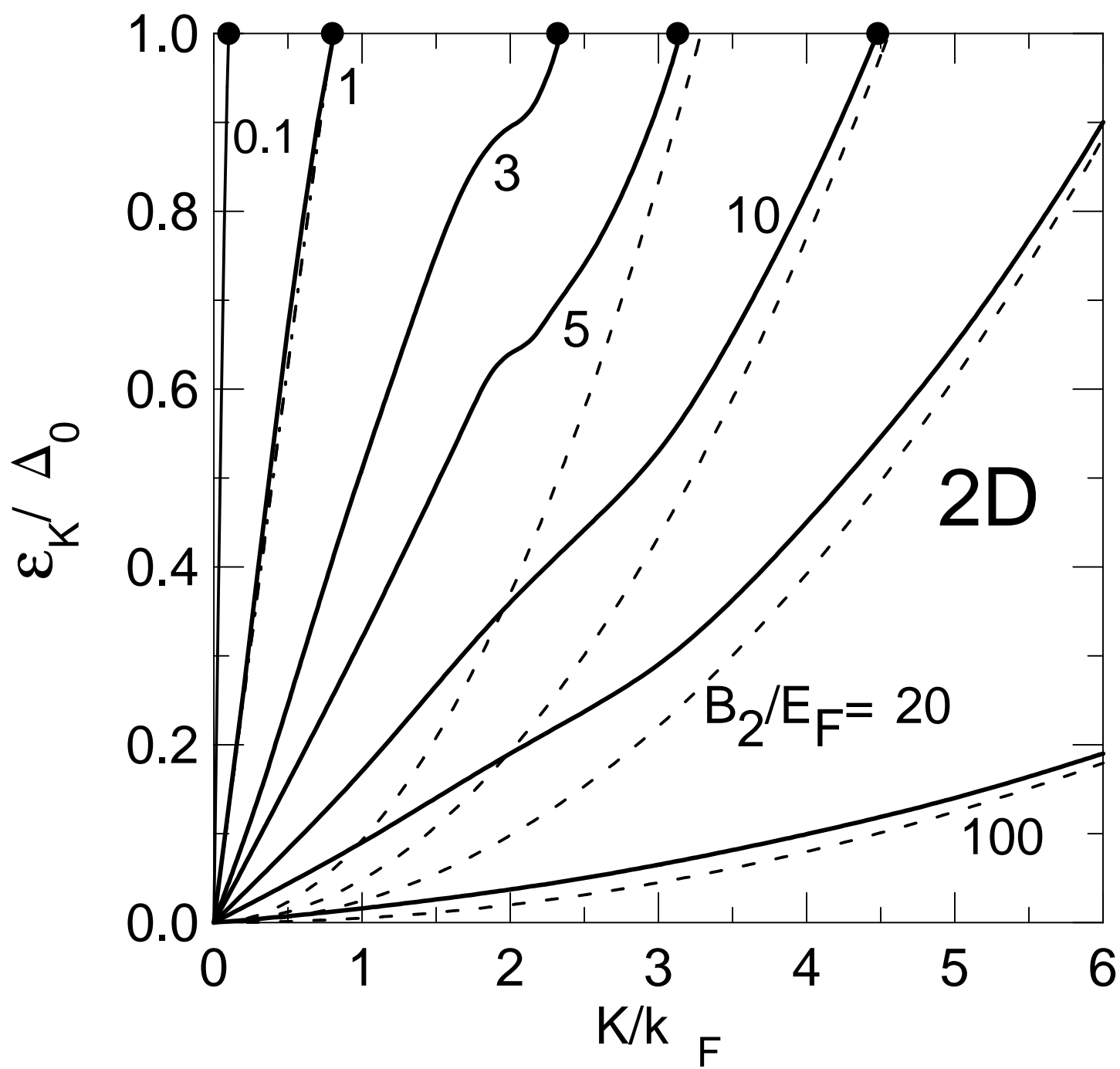
since $\zeta(3/2) \simeq 2.612$. Recalling that $k_B T_F = \hbar^2 k_F^2 / 2m$ with $k_F = [2^{d-2} \pi^{d/2} d\Gamma(d/2)n]^{1/d}$ from (36), then if $m_B = 2m$ and $n_B = n/2$ (all electrons paired) for $s = 2$ (C.1) implies that

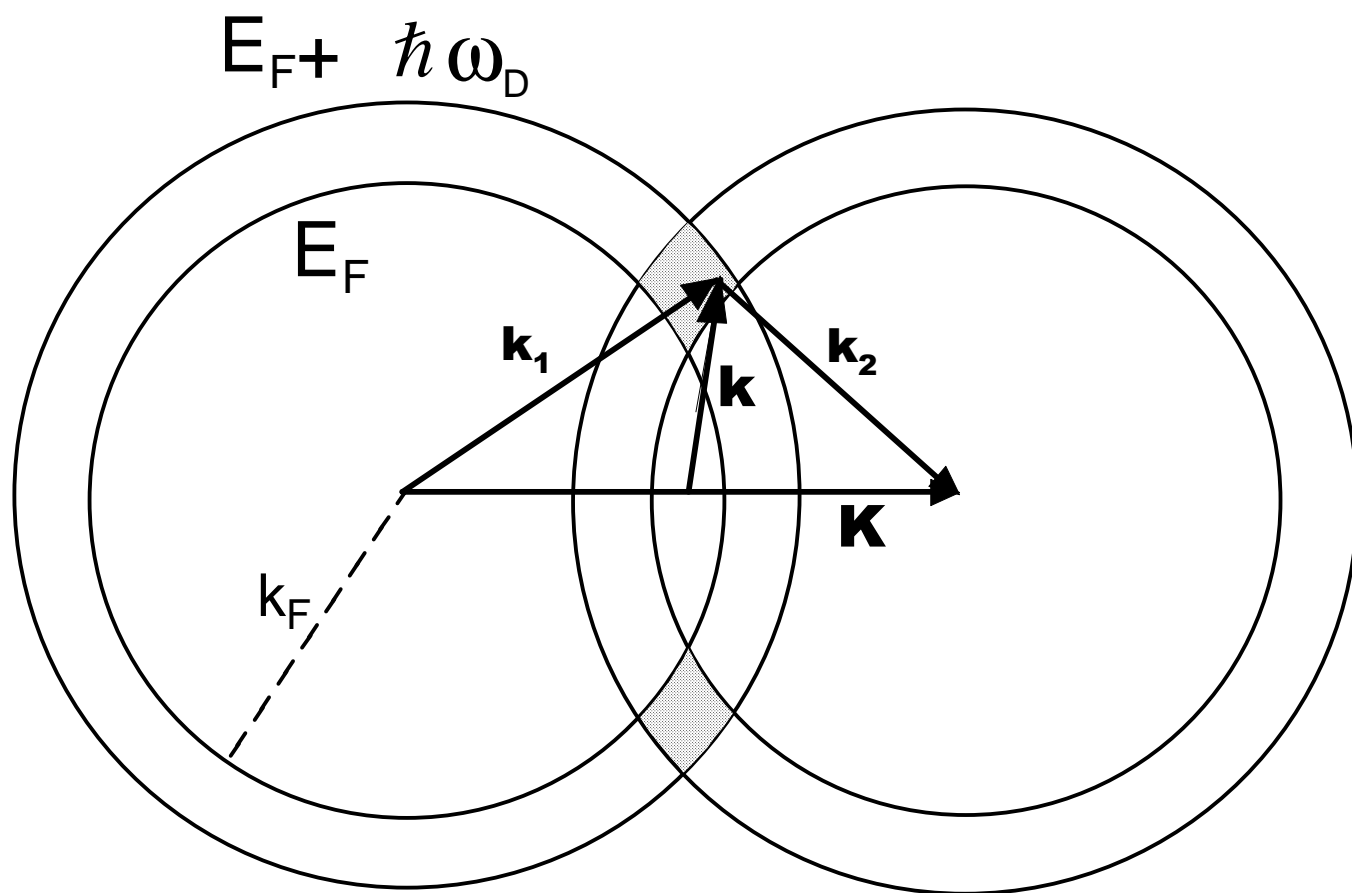
$$T_c/T_F = \frac{1}{2} [2/d\Gamma(d/2)g_{d/2}(1)]^{2/d} = 0 \quad \text{for} \quad d \leq 2 \quad (\text{C.3})$$

since $g_{d/2}(1) = \infty$ for $d/2 \leq 1$. When $d > 2$ then T_c is nonzero. For $d = 3$ we get the familiar limit mentioned before

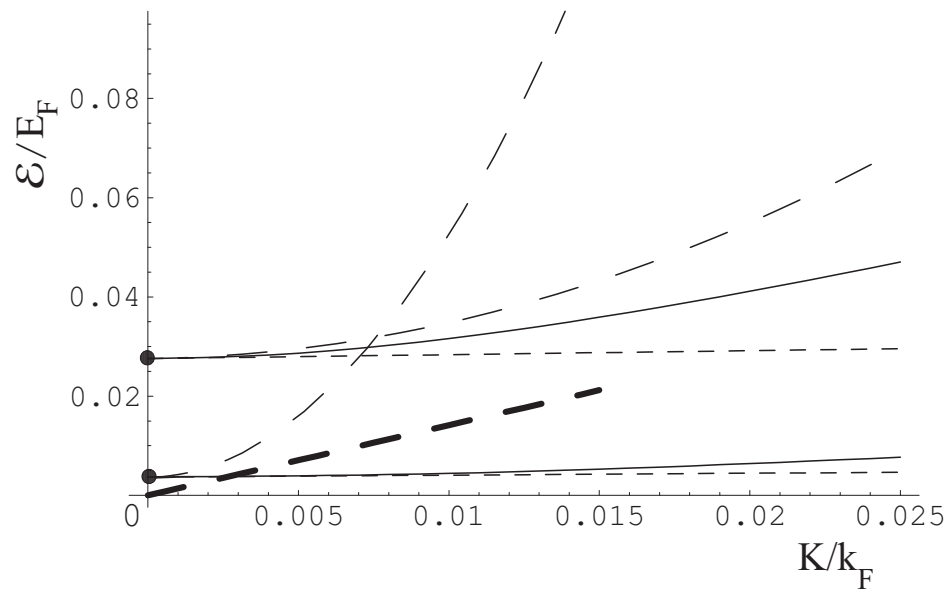
$$T_c/T_F = \frac{1}{2} [2/3\Gamma(3/2)\zeta(3/2)]^{2/3} \simeq 0.218, \quad (\text{C.4})$$

(see also dashed line Fig. 2 of Ref. [118]).

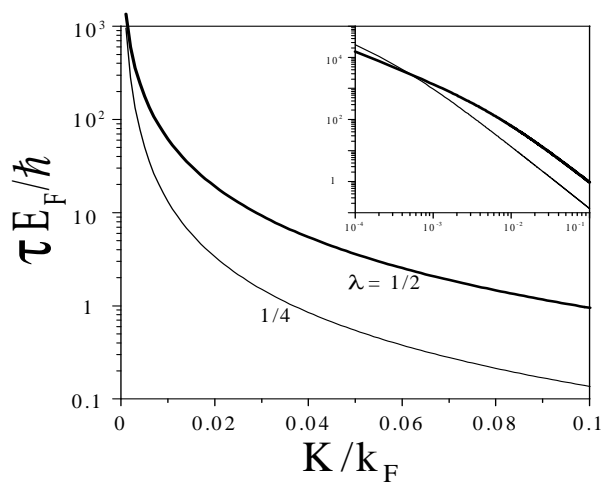




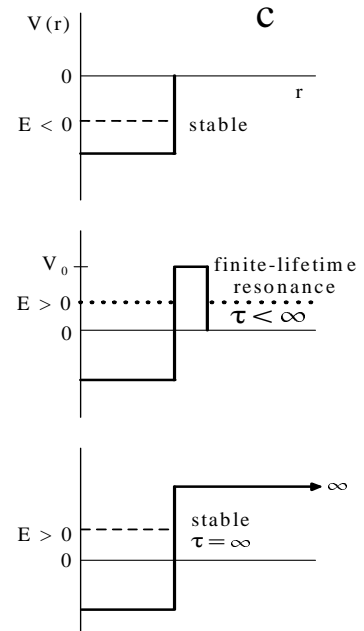
a

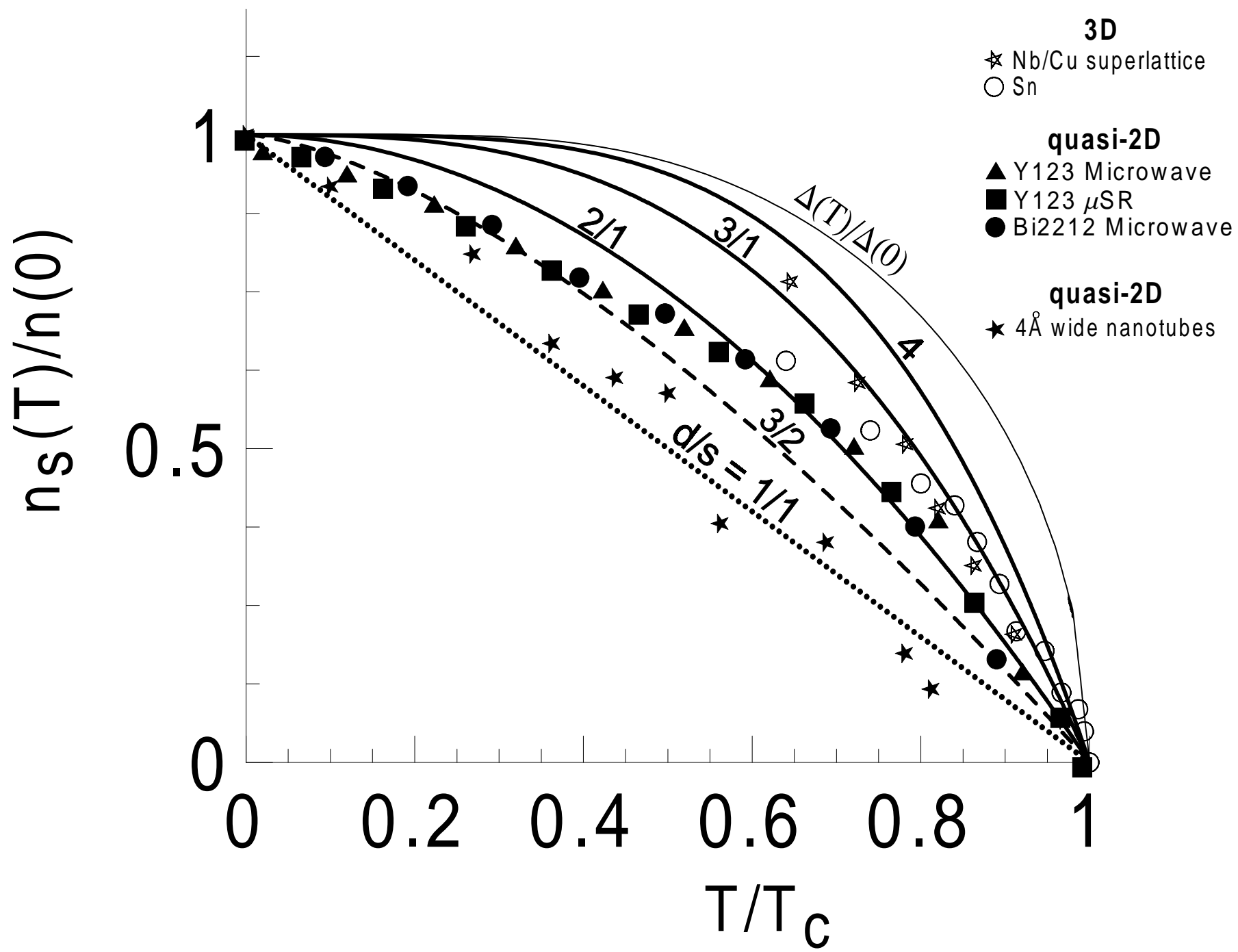


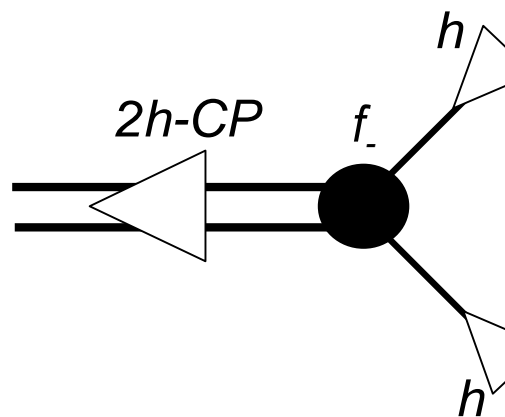
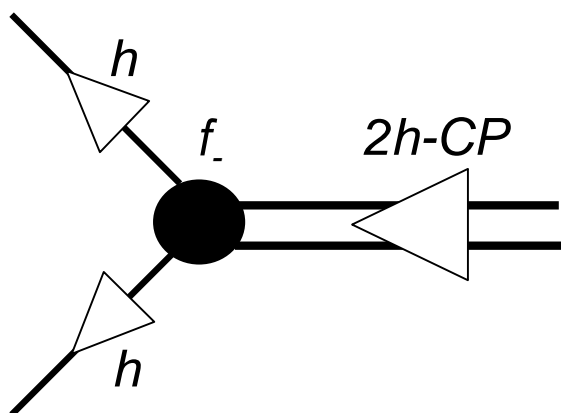
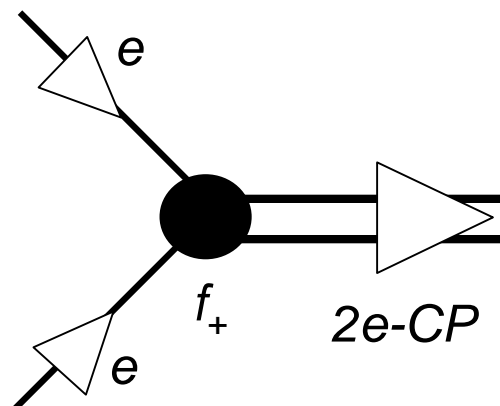
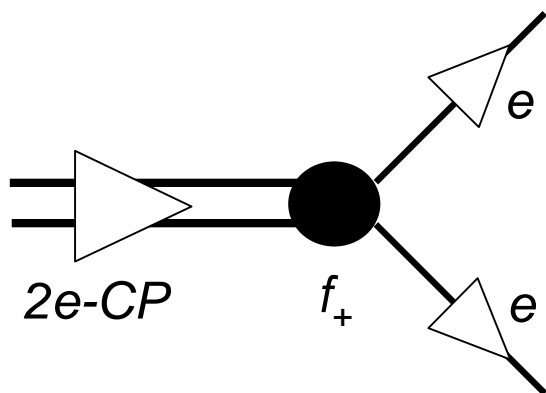
b



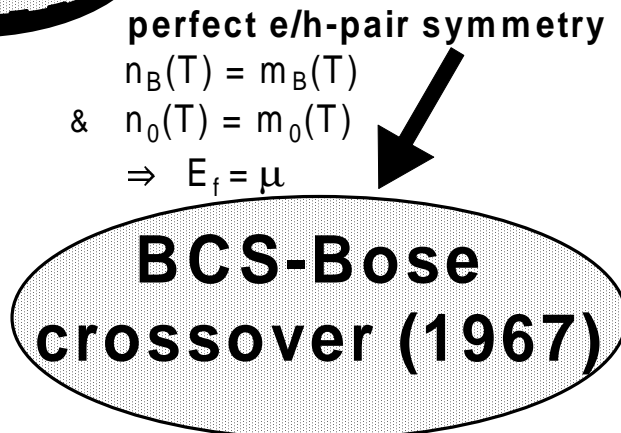
c



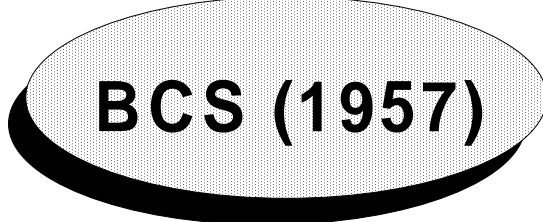




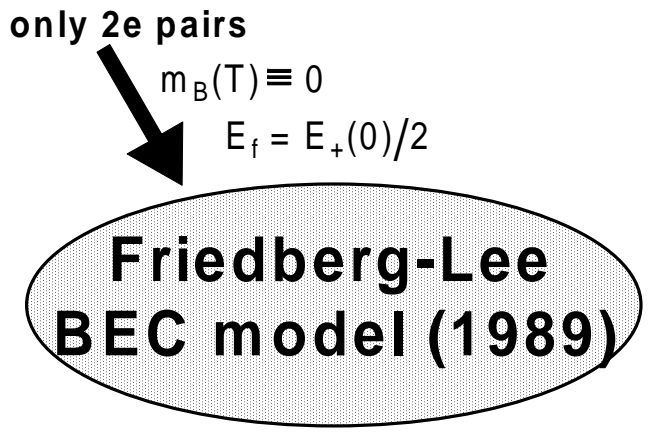
$$H = H_0 + H_{\text{int}} \equiv H_{e/h} + H_{e/h \text{ pairs}} + \left\{ \begin{array}{c} e \\ \nearrow \\ f_+ \\ \bullet \\ \searrow \\ e \end{array} \begin{array}{c} f_+ \\ 2e \text{ pairs} \end{array} \dots \right\}$$



(1 gap + 1 number eqn)
 $f = \sqrt{2\hbar\omega_b V} \rightarrow 0$
 $\Rightarrow \mu \rightarrow E_F$



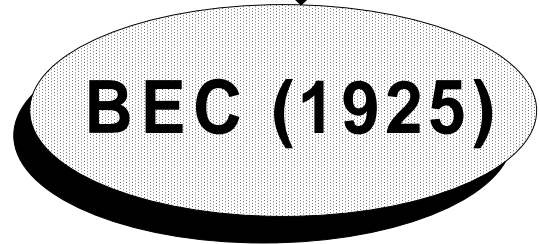
(T-dependent gap eqn)
 $\Rightarrow 2\Delta(0)/k_B T_c \simeq 3.53$



$\Delta(T) = f\sqrt{n_{B,0}(T)}$
 $f = 0 \Rightarrow \mu = E_f$



ideal BF model
 $[n - n_f(T_c)]/2 \equiv n_B(T_c)$



$T_c \simeq 3.31 \hbar^2 n_B(T_c)^{2/3} / 2mk_B$

If $n \gg n_f(T_c)$: $T_c/T_F = \frac{1}{2} [2/3 \Gamma(3/2) \zeta(3/2)]^{2/3} \simeq 0.218$

